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PRE-GENERALIZED-REGULAR AND STRONGLY-NORMAL SPACES

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The purpose of this paper is to study the classes of regular spaces & normal spaces, namely *pg*-regular spaces & *pq*-normal spaces which are a generalization of the classes of *p*-regular & *p*-normal spaces respectively. We, first, introduce regular spaces in terms of above mentioned nearly open set (*i.e.* *p*-open set) one by one & highlight the related characterizations of such spaces. It also focuses some basic properties and preservation criteria.

KEY WORDS : *M*-pre open set, *pg*-regular, strongly-normal and pre-quasi -normal space.

INTRODUCTION & PRELIMINARIES

In general topology, the notion of pre-open set, introduced by A.S. Mashour *et al.* [1982], has a significant role and the most important generalizations of regularity and normality appear as the notions of pre-regularity along with strong regularity [1983] and pre-normality as well as strong normality [1984] respectively.

The present paper consists of the generalization of the concepts of a regular space as well as a normal space in terms of *p*-open set.

Various new topological concepts and their basic properties have been defined and investigated using the notion of pre-open sets and pre-open, pre-continuous mappings (*i.e.* pre homeomorphism) as introduced by A.S. Mashhour *et al.* [1]. In 1998, T. Noiri *et al.* [2] studied generalized pre closed functions using generalized preclosed sets.

A subset A of a space (X, T) is known as a generalized pre-closed iff every open superset of A contains its pre-closure [2].

Since, many topologists have utilized these concepts to the various notions of subsets, weak separation axioms, weak regularity, and weaker and stronger forms of covering axioms in the literature. In this paper we utilize these sets to define and study the new classes of spaces, called p-regular spaces in topology. Also we characterize their basic properties along with already existing weaker forms of regularity. The characterization as well as the preservation theorems for pre quasi-normal spaces with common basic properties has been focused and prepared as a ready reckoner for the researchers in this paper.

In the present paper, spaces (X, T) and (Y, σ) always mean topological spaces which are not assumed to satisfy any separation axioms are assumed unless explicitly mentioned.

Also, $f : (X, T) \rightarrow (Y, \sigma)$ denotes a single valued function f of a space (X, T) into another space (Y, σ). And for a subset A of a space (X, T), $X/A = A^c$, cl (A) and int (A) denote the complement, the closure and the interior of A in (X, T) respectively

Definition (1.1) : A subset A of a topological space (X, T) is called

I. regular open or open domain [3] if $A = \text{int}(\text{cl}(A))$.

II. an α-open set [6] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$

III. pre-open or nearly open [1] set if $A \subseteq \text{int}(\text{cl}(A))$

The compliments of the above mentioned open sets are their respective closed sets. The smallest $\mathcal{K}$-closed set containing A is called $\mathcal{K}$cl (A) where $\mathcal{K}$ = regular, α, p. The largest $\mathcal{K}$-open set contained in A is called $\mathcal{K}$int (A) where $\mathcal{K}$ = regular, α, p.

The family of all $\mathcal{K}$-open (resp. $\mathcal{K}$-closed) sets of a space (X, T) is denoted by $\mathcal{K}O(X)$ (resp. $\mathcal{K}C(X)$); here and above $\mathcal{K}$ = regular, α, p.

Definition (1.2) [1, 3] : (a) The p-interior of a subset A, denoted as p-int (A), is defined as the union of all p-open sets contained in A.

(b) The pre-closure of subset A, denoted as $p - \text{cl}(A)$, is defined as the intersection of all p-closed sets containing A. Naturally, p-int (A) is pre open where as pcl (A) is pre closed where A is subset of X.

Also, pint $(A) = A \cap (\text{int}(\text{cl}(A))$ and pcl $(A) = A \cup \text{cl}(\text{int}(A))$.

Definition (1.3) [2] : A subset A of a space (X, T) is said to be generalized preclosed (briefly *gp*-closed) iff $p\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

PRE REGULAR SPACES:

The concepts of a regular space are generalized in terms of pre-open sets in the following two ways:

Definition (2.1): A topological space (X, T) is said to be pre-regular (in short p-regular) space iff every closed set F and every point $x \notin F$, there exists pre-open sets U and V such that

$$x \in U, F \subset V \text{ and } U \cap V = \varphi.$$

Definition (2.2) [9] : A topological space (X, T) is said to be pre- generalized regular (in short *pg*-regular) space iff every pre-closed set F and every point $x \notin F$, there exists pre-open sets U and V such that

$$x \in U, F \subset V \text{ and } U \cap V = \varphi.$$

Example (2.3) : This example indicates that the above concepts are different .

We consider the topological space (X, T) where $X = \{a, b, c, d\}$ and $T = \{\varphi, \{b\}, \{c\}, \{b, c\}, X\}$

The closed subset of this space are :

$$\varphi, \{a, d\}, \{a, c, d\}, \{a, b, d\}, X$$

Pre-open subsets of this space are :

$$\varphi, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X$$

So, pre- closed subset of X are :

$$\varphi, \{a\}, \{d\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}, X.$$

We observe that (X, T) is pre- regular space but it is not pre-generalized regular space. Also (X, T) is not a regular space.

Factual Observations:

(1) (X, T) is p-regular $\not\Rightarrow$ (X, T) is pg-regular.

(2) (X, T) is p-regular $\not\Rightarrow$ (X, T) is regular.

(3) Every regular space is p-regular.[9]

(4) Every pre generalized regular space is p-regular.[9]

(5) A pre regular space need not be a T_1 space. As seen by the next example:

Consider the topology $T = \{X, \varphi, \{a\}, \{b, c\}\}$ on the set $X = \{a, b, c\}$.

Then $PO(X, T) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X, \varphi\}$. Observe that the closed subset of X are also $\varphi, X, \{a\}, \{b, c\}$ and that (X, T) is pre-regular. On the other hand (X, T) is not a T_1 space.

Characterization Theorem (2.4) :

A topological space (X, T) is said to be pre-generalized regular iff for every $x \in X$ and for every pre-open set U containing x there exists a pre open set V such that $x \in V \subseteq \text{pcl } V \subseteq U$.

Proof : Let (X, T) be a be pre-generalized regular space. Let $x \in X$ and let U be a pre-open set containing x. Then U^c is pre-closed and $x \notin U^c$. Since X is pre-generalized regular, there exists a pre open sets V and W such that $x \in V$, $U^c \subset W$ and $V \cap W = \varphi$.

And this means that $x \in V$, $W^c \subset U$.

Since, W^c is pre closed, $x \in V \subseteq \text{pcl } V \subseteq W^c \subset U$.

Conversely, suppose that the given condition is satisfied.

Let F be pre closed and let $x \notin F$. Then F^c is pre open and $x \in F^c$. So according to the given condition there exists a pre-open set V such that $x \in V \subseteq \text{pcl } V \subseteq F^c$.

Now, $(\text{pcl } V)^c$ is a pre-open set such that $x \in V$, $F \subseteq (\text{pcl } V)^c$ and $V \cap (\text{pcl } V)^c = \varphi$.

This establishes that (X, T) is a pre-generalized regular space.

Theorem (2.5) [9] : A topological space (X, T) is said to be pre- regular iff for every $x \in X$ and for every open set U containing x there exists a pre open set V such that $x \in V \subseteq \text{pcl } V \subseteq U$.

Proof : The result can be proved in the similar manner with proper changes according to the context.

Preservation criteria :

Definition (2.6) : A bijection $f : (X, T) \to (Y, \sigma)$ is called pre-homeomorphism if f is both pre-continuous and pre-open map.

Theorem (2.7) : If $f : (X, T) \to (Y, \sigma)$ is a bijection, pre-homeomorphism and (X, T) is regular space then Y is a pre-regular space.

Proof : Let $y \in Y$ and F be any closed set in Y such that $y \notin F$. Since f is bijection, there exists a point $x \in X$ such that $f(x) = y \Rightarrow x = f^{-1}(y)$. Again since f is pre-continuous, $f^{-1}(F)$ is a pre-closed set in X. Also $y \notin F \Rightarrow f^{-1}(y) \notin f^{-1}(F) \Rightarrow x \notin f^{-1}(F)$. Since X is regular space, there exist open sets M, N in X such that $x \in M$, $f^{-1}(F) \subset N$ and $M \cap N = \phi$. Since f is pre-open map, $f(M)$, $f(N)$ are pre-open sets in Y. Now we have $x \in M \Rightarrow f(x) \in f(M) \Rightarrow y \in f(M)$; $f^{-1}(F) \subset N \Rightarrow f[f^{-1}(F)] \subset f(N) \Rightarrow F \subset f(N)$ and $M \cap N = \phi \Rightarrow f(M) \cap f(N) = f(\phi) = \phi$, since f is a bijection. Thus, for every point $y \in Y$ and each closed set F in Y such that $y \notin F$, there exist pre-open sets $f(M)$, $f(N)$ in Y such that $y \in f(M)$, $F \subset f(N)$ and $f(M) \cap f(N) = \phi$. Hence, (Y, σ) is a pre-regular space. Hence, the theorem.

Theorem (2.8) : If $f : (X, T) \rightarrow (Y, \sigma)$ is a bijection, pre-irresolute, pre-closed map and (Y, σ) is pre-regular space then (X, T) is also pre-regular space.

Proof: Let $x \in X$ and F be any pre-closed set in X such that $x \notin F$. Since f is bijection, there exists a point $y \in Y$ such that $f(x) = y \Rightarrow x = f^{-1}(y)$. Also since f is closed map, $f(F)$ is a pre-closed set in Y such that $x \notin F \Rightarrow f(x) \notin f(F) \Rightarrow y \notin f(F)$. Since Y is pre-regular space, there exist pre-open sets M, N in Y such that $y \in M$, $f(F) \subset N$ and $M \cap N = \phi$. Since f is pre-irresolute, $f^{-1}(M)$, $f^{-1}(N)$ are pre-open sets in X. Now we have $y \in M \Rightarrow f^{-1}(y) \in f^{-1}(M)$ $\Rightarrow x \in f^{-1}(M)$; $f(F) \subset N \Rightarrow f^{-1}[f(F)] \subset f^{-1}(N) \Rightarrow F \subset f^{-1}(N)$ and $f^{-1}(M \cap N) = f^{-1}(\phi)$ $\Rightarrow f^{-1}(M) \cap f^{-1}(N) = \phi$, since f is a bijection. Thus, for every point $x \in X$ and each closed set F in X such that $x \notin F$, there exist pre-open sets $f^{-1}(M)$, $f^{-1}(N)$ in X such that $x \in f^{-1}(M)$, $F \subset f^{-1}(N)$ and $f^{-1}(M) \cap f^{-1}(N) = \phi$. Hence (X, T) is a pre-regular space. Hence, the theorem.

PRE-NORMAL SPACES

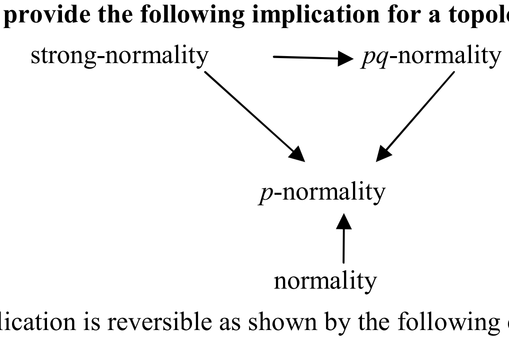
The concepts of a normal space are generalized in terms of pre-open sets in the following ways:

Definition (3.1) [9] : A space (X, T) is said to be pre-normal or p-normal if for each pair of disjoint closed sets A and B of X there exist pre-open sets U and V for which $A \subseteq U$ and $B \subseteq V$ such that $U \cap V = \varphi$.

Definition (3.2) : A space (X, T) is said to be strongly-normal or pre-generalized normal (briefly, stg-normal or pg-normal) if for each pair of disjoint pre-closed sets A and B of X there exist pre-open sets U&V in the manner $A \subseteq U$ and $B \subseteq V$ such that $U \cap V = \varphi$.

Definition (3.3) : A space (X, T) is said to be pre-quasi -normal (briefly, pq-normal) if for each pair of disjoint closed set A and pre- closed set B of X there exist pre-open sets U and V in the manner $A \subseteq U$ and $B \subseteq V$ such that $U \cap V = \varphi$.

The above definitions provide the following implication for a topological space (X, T) :



None of these implication is reversible as shown by the following examples:

Example (3.4) : (i) $X = \{a, b, c, d\}$, $T = \{\Phi, \{a, d\}, \{a, b, d\}, \{a, c, d\}, X\}$, then, T^c = set of all T-closed sets = $\{\Phi, \{b\}, \{c\}, \{b, c\}, X\}$

Thus, simple computations show that p-open sets in (X, T) are given by Φ, $\{a\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, X.

Obviously (X, T) is p-normal but not normal. But it is pg-normal as well as pq-normal.

(ii) Let $X = \{a, b, c\}$, $T = \{\Phi, \{b\}, \{b, c\}, X\}$, then T^c = set of all T-closed sets = $\{\Phi, \{a\}, \{a, c\}, X\}$.

Now, $PO(X, T) = \{\varphi, \{b\}, \{a, b\}, \{b, c\}, X\}$ & $PC(X, T) = \{\varphi, \{a\}, \{c\}, \{a, c\}, X\}$.

Obviously, (a) (X, T) is p-normal.

(b) (X, T) is not pg-normal.

(c) (X, T) is also not pq-normal.

(iii) Let $X = \{a, b, c\}$, $T = \{\Phi, \{a\}, \{a, b\}, X\}$, then, T^c = set of all T-closed sets = $\{\Phi, \{c\}, \{b, c\}, X\}$

Now, $PO(X, T) = \{\varphi, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $PC(X, T) = \{\varphi, \{b\}, \{c\}, \{b, c\}, X\}$.

Then (X, T) is p-normal but neither pg-normal nor pq-normal.

(iv) Let $X = \{a, b, c, d\}$, $T = \{\Phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$, then T-closed sets are: Φ, $\{b\}$, $\{a, b\}$, $\{c, d\}$, $\{b, c, d\}$, X.

Simple computations provide the following p-open sets:

Φ, $\{a\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, X.

This means that p-closed sets are given as:

Φ, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{c, d\}$, $\{b, c\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{b, c, d\}$, X.

All the implications, here, hold good.

The charecterization of p-normality is being thrown in the form of the following theorem :

Characterization Theorem (3.5) :

For a space (X, T) the following are equivalent:

(a) (X, T) is p-normal,

(b) For every pair of open sets U and V whose union is X, there exist p-closed sets A and B such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = X$.

(c) For every closed set H and every open set K containing H, there exists a p-open set U such that $H \subseteq U \subseteq p\text{-cl}(U) \subseteq K$.

Proof. (a) $\Rightarrow$ (b) : Let U and V be a pair of open sets in a p-normal space (X, T) such that $X = U \cup V$. Then U^C and V^C are disjoint closed sets. *i.e.*, $U^C \cap V^C = \varphi$. Since (X, T) is p-normal there exist disjoint p-open sets U_1 and V_1 such that $U^C \subseteq U_1$ and $V^C \subseteq V_1$.

Let $A = U_1^c$ and $B = V_1^c$. Then A and B are p-closed sets such that $A \subseteq U$, $B \subseteq V$. Also, $U_1 \cap V_1 = \varphi \Rightarrow U_1^c \cup V_1^c = X \Rightarrow A \cup B = X$.

(b) $\Rightarrow$ (c) : Let H be a closed set and K be an open set containing H. Now $H \subseteq K \Rightarrow K^C \subseteq H^C \Rightarrow K^C \cap H = \varphi \Rightarrow K \cup H^C = X$. Thus, H^c and K are open sets whose union is X.

Then by (b), there exist p-closed sets M_1 and M_2 such that $M_1 \subseteq H^c$ and $M_2 \subseteq K$. Then $H \subseteq M_1^c$, $K^c \subseteq M_2^c$ and $M_1^c \cap M_2^c = \varphi$.

Let $U = M_1^c$ and $V = M_2^c$. Then U and V are disjoint p-open sets such that $H \subseteq U \subseteq V^c \subseteq K$. As V^c is p-closedset, we have $p\text{-cl}(U) \subseteq V^c$ and $H \subseteq U \subseteq p\text{-cl}(U) \subseteq K$.

(c) ⇒ (a) : Let H_1 and H_2 be any two disjoint closed sets of X. Let $K = H_2^c$, then $H_2 \cap K = \varphi$. Now, $H_1 \subseteq K$, where K is an open set. Then by (c), there exists a p-open set U of X such that $H_1 \subseteq U \subseteq p\text{-cl}(U) \subseteq K$. It follows that $H_2 \subseteq [p\text{-cl}(U)]^c = V$, say, then V is p-open and $U \cap V = \varphi$. Hence H_1 and H_2 are separated by p-open sets U and V. Therefore (X, T) is p-normal.

Hence, the theorem.

Hereditary Criteria : We, now, highlight the specific hereditary property posses by a p-normal space.

Theorem (3.6) : A regular closed subspace of a p-normal space is p-normal.

Proof : Let (Y, T_Y) be a regular closed subspace of p-normal space (X, T).

Let A and B be disjoint closed subsets of Y. As Y is regular closed, hence closed, and consequently, A and B are closed subsets of X. Since A and B are closed subsets of X, so, by p-normality of X, there exist disjoint p-open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$. As every regular closed set is semi-open, by (lemma (2.2 [10]), $U \cap Y$ and $V \cap Y$ are pre-open in Y such that $A \subseteq U \cap Y$ and $B \subseteq V \cap Y$. Hence, (Y, T_Y) is p-normal. Hence, the theorem.

Remark: Pre-normality (p-normality) is hereditary with respect to regular closed set.

***p*-Topological property :**

Theorem (3.7) : p-normality is a p-topological property under a bijective, M-pre open and continuous mapping.

Proof : Let (X, T) be a p-normal space and $f : (X, T) \to (Y, \sigma)$ be a bijective, M-pre open and continuous mapping.

Suppose that A and B are two disjoint closed subset of Y. As f is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are two disjoint closed subsets of X. Since, (X, T) is p-normal, there exist disjoint pre-open sets U and V such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$.

Now, $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V \Rightarrow A \subseteq f(U)$ and $B \subseteq f(V)$. As f is M-pre open, so $f(U)$ and $f(V)$ are pre-open sets in Y such that $f(U) \cap f(V) = \varphi$. Accordingly, (Y, σ) is p-normal.

Theorem (3.8) : pg-normality is not a p-topological property under a bijective, M-pre open and p-continuous mapping.

Proof : Let (X, T) be a pg-normal space and $f : (X, T) \to (Y, \sigma)$ be a bijective, M-pre open and p-continuous mapping.

Suppose that A and B are two disjoint closed subset of Y. As f is p-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are two disjoint p-closed subsets of X. Since, (X, T) is pg-normal, there exist disjoint pre-open sets U and V such that

$$f^{-1}(A) \subseteq U \text{ and } f^{-1}(B) \subseteq V.$$

Now, $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V \Rightarrow A \subseteq f(U)$ and $B \subseteq f(V)$. As f is M-pre open, so $f(U)$ and $f(V)$ are pre-open sets in Y such that $f(U) \cap f(V) = \varphi$. Accordingly, (Y, σ) is p-normal.

Hence, the theorem.

The characterization of strongly normality or pre-generalized normality has been discussed in the following theorem:

Theorem (3.9): A topological space (X, T) is stg-normal or pg-normal iff every pre-closed set A and a pre-open set V containing A there is a pre-open set U such that $A \subseteq U \subseteq p\text{-cl}(U) \subseteq V$.

Proof : Let (X, T) be a *pg*-normal space. Let A be a pre-closed set in X and V be a pre-open set containing A then V^C is pre-closed and $A \cap V^C = \varphi$. Since, (X, T) is pg-normal, there exist pre-open sets U and W such that $A \subseteq U$ and $V^C \subseteq W$ where $U \cap W = \varphi$.

This means that $A \subseteq U \subseteq W^C \subseteq V$. Since, W^C is-pre-closed set containing the pre open set U, we have $U \subseteq p\text{-cl}(U) \subseteq W^C$. Thus $A \subseteq U \subseteq p\text{-cl}(U) \subseteq V$.

Conversely, suppose that the given condition is satisfied. Let A and B be disjoint pre-closed sets in X, then $A \cap B = \varphi \Rightarrow A \subseteq B^C$ and B^C is pre-open.

So there exists a pre-open set U such that $A \subseteq U \subseteq p\text{-cl}(U) \subseteq B^C$. This gives that $A \subseteq U$ and $B \subseteq [p\text{-cl}(U)]^C$ where $U \cap [p\text{-cl}(U)]^C = \varphi$. Hence, (X, T) is pg-normal.

Hence, the theorem.

Hereditary criteria : The specific hereditary property of a *pg*-normal space is mentioned as:

Theorem (3.10) : An open *gp*-closed subspace of a *pg*-normal space is *pg*-normal.

Proof : In order to establish the theorem we will have the use of the following lemmas :

Lemma (I) : If $A \subseteq Y \subseteq X$ and $A \in PO(X)$, then $A \in PO(Y)$ whenever Y is open in X.

Proof : By hypothesis, $A \subseteq \text{Int Cl}(A)$. Therefore, $A \subseteq \text{Int Cl}\, A \cap Y = \text{Int Cl}\, A \cap \text{Int}\, Y = \text{Int}(\text{Cl}(A) \cap Y) = \text{Int Cl}_Y\, A \subseteq \text{Int}_Y \text{Cl}_Y A$. Therefore, $A \in PO(Y)$.

Lemma II : If $Y \subseteq X$ is open and $A \in PO(X)$, then $A \cap Y \in PO(Y)$.

Proof : Given that $A \in PO(X)$ and Y is an open set in X. Then, $A \cap Y \subseteq \text{Int Cl}(A) \cap Y = \text{Int Cl}(A) \cap \text{Int}\, Y = \text{Int}((\text{Cl}(A)) \cap Y) \subseteq \text{Int Cl}(A \cap Y)$. This implies that $A \cap Y \in PO(X)$. Then, by Lemma-I above, $A \cap Y \in PO(Y)$.

Lemma III : If $A \in PC(Y)$ and $Y \in PC(X)$, then $A \in PC(X)$.

Proof : Let $A \in PC(Y)$ implies that $\text{Cl}_Y \text{Int}_Y A \subseteq A$ and $Y \in PC(X)$ implies that, $\text{Cl}_X \text{Int}_X Y \subseteq Y$. Since $A \subseteq Y$ implies $\text{Int}\, A \subseteq \text{Int}\, Y$, $\text{Cl Int}\, A \subseteq \text{Cl Int}\, Y \subseteq Y$. Then, $\text{Cl}_Y(\text{Int}\, A) = [\text{Cl Int}\, A] \cap Y = \text{Cl Int}\, A$.

Also, $\text{Int}\, A \subseteq \text{Int}_Y A$ and hence, $\text{Cl Int}\, A \subseteq \text{Cl}(\text{Int}_Y A)$. Thus, we have $[\text{Cl Int}(A)] \cap Y \subseteq [\text{Cl Int}_Y A] \cap Y = \text{Cl}_Y \text{Int}_Y A$. Therefore, $\text{Cl Int}\, A \subseteq A$ which implies $A \in PC(X)$.

Proof of main theorem :

As a recall a subset A of (X, T) is said to be *gp*-closed iff $p\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

Suppose (Y, T_Y) is an open *gp*-closed subspace of a strongly normal space *i.e. pg*-normal space (X, T). Then, $p\text{Cl}_X Y = Y$, thus Y is preclosed set in X. Now, let A and B be any pair of disjoint preclosed sets in Y. Then, A and B are disjoint preclosed sets in X by Lemma-III. Since X is strongly normal, there exist disjoint preopen sets U and V such that $A \subseteq U$ and $B \subseteq V$. Again, $Y \cap U$ and $Y \cap V$ are preopen sets in Y by Lemma-II above. Hence, $A \subseteq Y \cap U$ and $B \subseteq Y \cap V$. Therefore, (Y, T_Y) is *pg*-normal space.

Hence, the theorem.

Remark : Strong-normality (*i.e. pg*-normality) is hereditary with respect to open *gp*-closed set.

Pre-topological property :

Theorem (3.11) : Strong normality is a pre-topological property *i.e.* pre homeomorphic image of a strongly normal space is a strongly normal space.

Proof : As a recall, a mapping $f : (X, T) \to (Y, \sigma)$ from a space (X, T) to another space (Y, σ) is known to be a pre-homeomorphishm iff f is one-one and onto, M-pre-open and pre-irresolute.

Let (X, T) be a strongly–normal (*i.e. pg*-normal) space and let (Y, σ) be its pre-homeomorphic image under the function f. Then f is bijective, M-preopen and pre-irresolute.

Now, we show that (Y, σ) is strongly–normal space :

Suppose E and F be two disjoint pre closed sets of Y. As f is pre irresolute, $f^{-1}(E)$ and $f^{-1}(F)$ be two disjoint pre closed sets in X. Since (X, T) is strongly normal, there exist disjoint pre open sets U and V such that $f^{-1}(E) \subseteq U$ and $f^{-1}(F) \subseteq V$. As $f^{-1}(E) \subseteq U$ implies $E \subseteq f(U)$ and $f^{-1}(F) \subseteq V$ implies $F \subseteq f(V)$. As f is M-pre open and bijective, $f(U), f(V) \in PO(Y)$ such that $f(U) \cap f(V) = \varphi$. This shows that (Y, σ) is strongly normal space.

Hence, the theorem.

The characterization of pre-quasi normality (*i.e. pq*-normality) has been enunciated as :

Theorem (3.12) : A topological space (X, T) is pq-normal iff every closed set A and a pre-open set V containing A there is a pre-open set U such that $A \subseteq U \subseteq p\text{-cl}(U) \subseteq V$.

Proof : The proof exists in the same tune as enunciated in the theorem (3.9) with proper changes according to the context of this theorem.

Hereditary criteria : Pre-quasi normality is not hereditary and may be hereditary with respect to some specified set, *e.g.* regular closed set, open *gp*-closed set etc.

Pre-topological property:

Theorem (3.13) : Pre-quasi normality is a pre-topological property *i.e.* pre-homeomorphic image of a pre-quasi generalized normal space is a pre-quasi generalized normal space.

Proof : The proof appears in the same manner as indicated in that of theorem (3.11) with suitable changes according to the context.

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