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LI-IDEAL IN COMMUTATIVE *l*-GROUP IMPLICATION ALGEBRA

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In this paper, the idea of Lattice implicative ideal or *LI*-ideal and quotient commutative *l*-group implication algebra are introduced, fundamental theorem of homomorphism, first isomorphism theorem, second isomorphism theorem are established. Further established set of all *LI*-ideals of a commutative *l*-group implication algebra form a distributive lattice.

KEYWORDS : Lattice implicative ideal or *LI*-ideal, quotient commutative *l*-group implication algebra, and distributive lattice.

Introduction

It is well known that a distributed complimented lattice is a Boolean Algebra which is equivalent to Boolean ring with identity. This relation gives a link between Lattice theory and Modern Algebra. The algebraic structure connecting lattice and group is called *l*-group or lattice ordered group. Many common abstractions, namely Dually residuated lattice ordered semi groups, lattice ordered commutative groups, lattice ordered near rings lattice ordered semi rings and commutative *l*-group implication algebra are presented in [6], [3], [1], [5] and [4] respectively. The concept of LI-ideal in lattice implication algebra is introduced in [7].

In this paper to introduced and established the study of LI-ideal in Commutative *l*-group implication algebra.

Preliminaries

In this section are listed a number of definitions and results which are made use of throughout the paper. The symbols $\leq, +, -, \vee, \wedge, \rightarrow, *$ and $\in$ will denote inclusion, sum, difference, join (least upper bound), meet (greatest lower bound), implication, symmetric difference and membership in a lattice L or commutative l-group implication algebra G. Small letters $a, b, \ldots$ will denote elements of the lattice L or commutative l-group G.

Definition 1.1 : A non-empty set G is called an l-group iff

(i) $(G, +)$ is a group

(ii) $(G, \leq)$ is a lattice

(iii) If $x \le y$, then $a + x + b \le a + y + b$, for all a, b, x, y in G.

or

$$(a + x + b) \vee \vee (a + y + b) = (a + x \vee \vee y + b)$$

$$(a + x + b) \wedge \wedge (a + y + b) = (a + x \wedge \wedge y + b), \quad \text{for all } a, b, x, y \text{ in } G.$$

Definition 1.2: An l-group G is called commutative l-group if $x + y = y + x$ for all x, y in G.

Definition 1.3:

An implication algebra is a non-empty set L with greatest element I, least element 0, an unary operation " ' " and a binary operation "→" which satisfies the following axioms:

(I1) $1 \to x = x$

(I2) $x \to x = \mathrm{I}$

(I3) $(x \to y) \to y = (y \to x) \to x$

(I4) $(((y \to z) \to z) \to x) \to x = (((y \to x) \to x) \to z) \to z$

(I5) $x \to (y \to z) = y \to (x \to z)$

(I6) $0 \to x = \mathrm{I}$

(I7) $x \to 0 = x'$ for all $x, y, z \in L$.

Definition 1.4: Let $(L, \vee\vee, \wedge\wedge, 0, I)$ be a bounded lattice with an order-reversing involution $'$, I and 0 the greatest and the smallest element of L respectively, $\to : L \times L \to L$ be a mapping. Then $(L, \vee\vee, \wedge\wedge, ', \to, O, I)$ is called a lattice implication algebra if the following conditions hold for any $x, y, z \in L$:

(L_1) $x \to (y \to z) = y \to (x \to z)$,

(L_2) $x \to x = \mathrm{I}$,

(L_3) $x \to y = y' \to x'$,

(L_4) If $x \to y = y \to x = I$, then $x = y$,

(L_5) $(x \to y) \to y = (y \to x) \to x$,

(L_6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

(L_7) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

The binary operation "→" will be denoted by juxt a position. We can define a partial ordering "≤" on a lattice implication algebra L by $x \le y$ if and only if $x \to y = 1$.

Theorem 1.1. Definitions 1.3 and 1.4 are equivalent.

Theorem 1.2. In a lattice implication algebra L, the following are hold

(i) $x \le y$ if and only if $x \to y = 1$

(ii) $x \le (x \to y) \to y$

(iii) $0 \to x = 1$, $1 \to x = x$ and $x \to 1 = 1$

(iv) $x' = x \to 0$

(v) $x \to y \le (y \to z)\ (x \to z)$

(vi) $(x \vee y) = (x \to y) \to y$

(vii) $x \le y \Rightarrow y \to z \le x \to z$ and $z \to x \le z \to y$.

Definition 1.5. A non-empty set G is called **commutative *l*-group implication algebra** if only if

1. $(G, +)$ is a commutative group
2. $(G, \rightarrow)$ is an implication algebra
3. $x \leq y \Longrightarrow$ (i) $a + x \leq a + y$
   (ii) $(a \rightarrow x) \rightarrow b \geq (a \rightarrow y) \rightarrow b$
   (iii) $a \rightarrow (x \rightarrow b) \geq a \rightarrow (y \rightarrow b)$, for all a, b, x, y in G.

Definition 1.6. A non empty set G is called **commutative *l*-group implication algebra** if and only if

1. $(G, +)$ is a commutative group
2. $(G, \rightarrow)$ is an implication algebra
3. (i) $a + (x \vee y) = (a + x) \vee (a + y)$
   (ii) $a + (x \wedge y) = (a + x) \wedge (a + y)$
   (iii) $[a \rightarrow (x \vee y)] \rightarrow b] = [(a \rightarrow x) \rightarrow b] \wedge [(a \rightarrow y) \rightarrow b]$
   $= a \rightarrow [(x \vee y) \rightarrow b]$
   (iv) $[a \rightarrow (x \wedge y)] \rightarrow b] = [(a \rightarrow x) \rightarrow b] \vee [(a \rightarrow y) \rightarrow b]$
   $= a \rightarrow [(x \wedge y) \rightarrow b]$ for all x, y, a, b in G.

Theorem 1.3. The above two definitions for commutative *l*–group implication algebra are equivalent.

LI-IDEAL IN COMMUTATIVE *L*-GROUP IMPLICATION ALGEBRA

In this section to introduced lattice implicative ideal or *LI*-ideal and obtained set of all *LI*-ideals in a commutative *l*-group implication algebra form a distributive lattice.

Definition 2.1: Let G be a commutative *l*–group implication algebra and I a non-empty subset of G. Then I is called an *LI*–ideal if and only if

1. a, b in I implies $a - b$ in I
2. a, b in I implies $a \vee b$, $a \wedge b$ in I
3. $0 < x < a$, and a in I implies x in I
4. $(x \rightarrow y)' \in I$ and $y \in I$ imply $x \in I$

In a commutative *l*–group implication algebra, $\{0\}$, G are *LI*–ideals of G.

Theorem 2.1: If I_1, I_2, are two *LI*–ideals of commutative *l*–group implication algebra G, then

(i) $I_1 \vee I_2 = \{ x \,\varepsilon\, G / x \leq x_1 \vee x_2$ for some x_1 in I_1, x_2 in $I_2\}$ is an *LI*–ideal

(ii) $I_1 \wedge I_2 = \{ x \,\varepsilon\, G / x$ in I_1 and x in $I_2\}$ is an *LI*–ideal

(iii) $I_1 + I_2 = \{ x$ in $G / x \leq x_1 + x_2$ for some x_1 in I_1, x_2 in $I_2\}$ is an *LI*–ideal

(iv) $I_1 \vee I_2$ is the smallest *LI*– ideal containing $I_1 \cup I_2$

Proof: For (i):

Let a, b in $I_1 \vee I_2$

$\Longrightarrow$ a, b in G such that $a \leq a_1 \vee a_2$, $b \leq b_1 \vee b_2$ for some a_1 , b_1 in I_1, a_2, b_2 in I_2

$\Longrightarrow$ $a - b$, $a \vee b$, $a \wedge b$ in G such that

$a - b \leq (a_1 \vee a_2) - (b_1 \vee b_2) = [a_1 - (b_1 \vee b_2)] \vee [(a_2 - (b_1 \vee b_2)]$

$= [(a_1 - b_1) \wedge (a_1 - b_2)] \vee [(a_2 - b_1) \wedge (a_2 - b_2)]$, by property (4) and (5)

$\leq$ $(a_1 - b_1) \vee (a_2 - b_2)$ with $a_1 - b_1$ in I_1, $a_2 - b_2$ in I_2.

$a \vee b \leq (a_1 \vee a_2) \vee (b_1 \vee b_2) = a_1 \vee [a_2 \vee (b_1 \vee b_2)] = a_1 \vee [(a_2 \vee b_1) \vee b_2]$

$= a_1 \vee [(b_1 \vee a_2) \vee b_2] = a_1 \vee [b_1 \vee (a_2 \vee b_2)] = (a_1 \vee b_1) \vee (a_2 \vee b_2)$ for some $a_1 \vee b_1$ in I_1, $a_2 \vee b_2$ in I_2

$a \wedge b \leq (a_1 \vee a_2) \wedge (b_1 \vee b_2) = [(a_1 \vee a_2) \wedge b_1] \vee [(a_1 \vee a_2) \wedge b_2]$

$= [b_1 \wedge (a_1 \vee a_2)] \vee [b_2 \wedge (a_1 \vee a_2)] = [(b_1 \wedge a_1) \vee (b_1 \wedge a_2)] \vee [b_2 \wedge a_1) \vee (b_2 \wedge a_2)]$

$\leq$ $(a_1 \vee b_1) \vee (a_2 \vee b_2)$ with $a_1 \vee b_1$ in I_1, $a_2 \vee b_2$ in I_2

$\Longrightarrow$ $a - b$, $a \vee b$, $a \wedge b$ in $I_1 \vee I_2$

Let $0 < x < a$ and a in $I_1 \vee I_2$

$\Longrightarrow$ $0 < x < a$ and a in G such that $a \leq a_1 \vee a_2$ for some a_1 in I_1, a_2 in I_2

$\Longrightarrow$ x in G such that $x \leq a_1 \vee a_2$ for some a_1 in I_1, a_2 in $I_2 \Longrightarrow x$ in $I_1 \vee I_2$

Let $(a \to b)' \in I_1 \vee I_2$, $b \in I_1 \vee I_2$

$\Longrightarrow b \to a \in I_1 \vee I_2$, $b \in I_1 \vee I_2$

$\Longrightarrow b \to a \in G$ such that $b \to a \leq c_1 \vee c_2$ for some $c_1 \in I_1$, $c_2 \in I_2$

$b \leq b_1 \vee b_2$ for some $b_1 \in I_1$, $b_2 \in I_2$

$a \leq b \to a \leq c_1 \vee c_2$ for some $c_1 \in I_1$, $c_2 \in I_2$

$\Longrightarrow a \in I_1 \vee I_2$

Thus $I_1 \vee I_2$ is an LI–ideal.

For (ii):

Let a, b in $I_1 \wedge I_2$

$\Longrightarrow a$, b in I_1 and a, b in $I_2 \Longrightarrow a - b$, $a \vee b$, $a \wedge b$ in I_1 and $a - b$, $a \vee b$, $a \wedge b$ in I_2

$\Longrightarrow$ $a - b$, $a \vee b$, $a \wedge b$ in $I_1 \wedge I_2$

Let $0 < x < a$ and a in $I_1 \wedge I_2$

$\Longrightarrow$ $0 < x < a$ and a in I_1 and a in $I_2 \Longrightarrow$ ($0 < x < a$ and a in I_1) and ($0 < x < a$ and a in I_2)

$\Longrightarrow x$ in I_1 and x in $I_2 \Longrightarrow x$ in $I_1 \wedge I_2$

Let $(a \to b)' \in I_1 \wedge I_2$, $b \in I_1 \wedge I_2$

$\Longrightarrow (a \to b)' \in I_1$ and $b \in I_1$, and $(a \to b)' \in I_2$ and $b \in I_2$

$\Longrightarrow a \in I_1$ and $a \in I_2 \Longrightarrow a \in I_1 \wedge I_2$

Hence $I_1 \wedge I_2$ is an LI – ideal

For (iii) :

Let a, b in $I_1 + I_2$

$\Longrightarrow$ a, b in G such that $a \leq a_1 + a_2$, $b \leq b_1 + b_2$, for some a_1, b_1 in I_1, a_2, b_2 in I_2

$\Longrightarrow$ $a - b$, $a \vee b$, $a \wedge b$ in G such that

$a - b \leq (a_1 + a_2) - (b_1 + b_2) = (a_1 - b_1) + (a_2 - b_2)$ with $a_1 - b_1$ in I_1, $a_2 - b_2$ in I_2

$a \vee b \leq (a_1 + a_2) \vee (b_1 + b_2) \leq [(a_1 \vee b_1) + (a_2 \vee b_2] \vee [(a_1 \vee b_1) + (a_2 \vee b_2)]$

$= (a_1 \vee b_1) + (a_2 \vee b_2)$ for some $a_1 \vee b_1$ in I_1, $a_2 \vee b_2$ in I_2

$a \wedge b \leq (a_1 + a_2) \wedge (b_1 + b_2) \leq (a_1 + a_2)$ for some a_1 in I_1, a_2 in I_2

$\Longrightarrow$ $a - b$, $a \vee$ b, $a \wedge b$ in $I_1 + I_2$

Let $0 < x < a$ and a in $I_1 + I_2$

$\Longrightarrow$ $0 < x < a$ and a in G such that $a \leq a_1 + a_2$ for some a_1 in I_1, a_2 in I_2

$\Longrightarrow$ $x < a_1 + a_2$ for some a_1 in I_1, a_2 in I_2

$\Longrightarrow$ x in $I_1 + I_2$

Let $(a \rightarrow b)' \in I_1 + I_2$, $b \in I_1 + I_2$

$\Longrightarrow$ $b \rightarrow a \in I_1 + I_2$, $b \in I_1 + I_2$

$\Longrightarrow$ $a < b \rightarrow a \leq c_1 + c_2$ for some $c_1 \in I_1$, $c_2 \in I_2$

$\Longrightarrow$ $a \in I_1 + I_2$

Hence $I_1 + I_2$ is an *LI*–ideal of G.

For (iv):

To prove

(1) $I_1 \cup I_2 \subset I_1 \vee I_2$

(2) If $I_1 \cup I_2 \subset I$ then $I_1 \vee I_2 \subset I$ for any *LI*-ideal I of G.

For (1) :

Let $x \in I_1 \cup I_2$ be arbitrary

$\Longrightarrow x \in I_1$ or $x \in I_2$ or $x \in I_1 \cap I_2$

Case (i) : Suppose $x \in I_1$

$\Longrightarrow x \leq x \vee 0$ with $x \in I_1$, $0 \in I_2$

$\Longrightarrow x \in I_1 \vee I_2$

Case (ii) : Suppose $x \in I_2$

$\Longrightarrow 0 \leq 0 \vee x$ with $0 \in I_1$, $x \in I_2$

$\Longrightarrow x \in I_1 \vee I_2$

Case (iii) : Suppose $x \in I_1 \cap I_2$

$\Longrightarrow$ $x \in I_1$ and $x \in I_2$

$\Longrightarrow$ $x \leq x \vee x$ with $x \in I_1$, $x \in I_2 \Longrightarrow x \in I_1 \vee I_2$

Hence in all cases $x \in I_1 \cup I_2 \Longrightarrow x \in I_1 \vee I_2$

$\Longrightarrow I_1 \cup I_2 \leq I_1 \vee I_2$

For (2) : Suppose $I_1 \cup I_2 \subset I$ for any *LI*-ideal I of G

Then we claim that $I_1 \vee I_2 \subset I$

Let $x \in I_1 \vee I_2$ be arbitrary

$\Rightarrow x \leq x_1 \vee x_2$ for some x_1 in I_1, x_2 in $I_2 \Rightarrow x \leq x_1 \vee x_2$, x_1 in $I_1 \cup I_2 \subset I$, x_2 in $I_1 \cup I_2 \subset I$

$\Rightarrow x \leq x_1 \vee x_2$, $x_1 \vee x_2$ in $I \Rightarrow x \in I$

Therefore $I_1 \vee I_2 \subset I$

Hence $I_1 \vee I_2$ is the smallest LI – ideal containing $I_1 \cup I_2$

Theorem 2.2: Let G be a commutative l–group implication algebra and $I(G)$, set of all LI–ideals of G. Then $I(G)$ is a lattice.

Proof : ⇔

First to claim that $(I(G), \leq)$ is a poset

Let I_1 , I_2 , I_3 in $I(G)$ be arbitrary.

Define a relation $\leq$ on $I(G)$ by

$$I_1 \leq I_2 \Leftrightarrow I_1 \subseteq I_2 \text{ where } I_1 , I_2 \text{ in } I(G).$$

'≤' is reflexive: $I_1 \leq I_1$, for all I_1 in $I(G)$

Then $I_1 \subseteq I_1 \Rightarrow I_1 \leq I_1$

Thus $I_1 \leq I_1$, for all I_1 in $I(G)$.

'≤' is anti symmetric: If $I_1 \leq I_2$ and $I_2 \leq I_1$ then $I_1 = I_2$ for all I_1, I_2 in $I(G)$

Suppose $I_1 \leq I_2$ and $I_2 \leq I_1 \Rightarrow I_1 \subseteq I_2$ and $I_2 \subseteq I_1 \Rightarrow I_1 = I_2$

Thus if $I_1 \leq I_2$ and $I_2 \leq I_1$ for all I_1 , I_2 in $I(G)$

'≤' is transitive: If $I_1 \leq I_2$ and $I_2 \leq I_3$ then $I_1 \leq I_3$, for all I_1, I_2, I_3 in $I(G)$

Suppose $I_1 \leq I_2$ and $I_2 \leq I_3$

$\Rightarrow I_1 \subseteq I_2$ and $I_2 \subseteq I_3 \Rightarrow I_1 \subseteq I_3 \Rightarrow I_1 \leq I_3$

Thus if $I_1 \leq I_2$ and $I_2 \leq I_3$ then $I_1 \leq I_3$ for all I_1, I_2, I_3 in $I(G)$.

Thus $(I(G), \leq)$ is a poset.

Next to claim that any two elements in $I(G)$ have a l.u.b and g.l.b in $I(G)$

$\Rightarrow I_1$, I_2 are LI–ideals of $G \Rightarrow I_1 \vee I_2$, $I_1 \wedge I_2$ are LI– ideals of G, by previous theorem

$\Rightarrow I_1 \vee I_2$, $I_1 \wedge I_2$ in $I(G)$ …(1)

We have $I_1 \subset I_1 \vee I_2$, $I_2 \subset I_1 \vee I_2 \Rightarrow I_1 \leq I_1 \vee I_2$, $I_2 \leq I_1 \vee I_2$ …(2)

Suppose I_3 is any other upper bound of I_1 and I_2 in $I(G)$

$\Rightarrow I_1 \leq I_3, I_2 \leq I_3 \Rightarrow I_1 \subset I_3 , I_2 \subset I_3 \Rightarrow I_1 \cup I_2 \subset I_3$

$\Rightarrow I_1 \vee I_2 \subset I_3$, since $I_1 \vee I_2$ is the smallest LI–ideal containing $I_1 \cup I_2$

Thus $I_1 \leq I_3$, $I_2 \leq I_3$ implies $I_1 \vee I_2 \leq I_3$ …(3)

From (1), (2) and (3), any two elements I_1 and I_2 in $I(G)$ have a l.u.b $I_1 \vee I_2$ in $I(G)$.

We have $I_1 \wedge I_2 \subset I_1$, $I_1 \wedge I_2 \subseteq I_2 \Rightarrow I_1 \wedge I_2 \leq I_1$, $I_1 \wedge I_2 \leq I_2$ …(4)

Suppose I_3 is any other lower bound of I_1 and I_2 in $I(G)$

$\Rightarrow I_3 \leq I_1$, $I_3 \leq I_2 \Rightarrow I_3 \subset I_1, I_3 \subseteq I_2 \Rightarrow I_3 \wedge I_3 \subset I_1 \wedge I_2 \Rightarrow I_3 \leq I_1 \wedge I_2$

Thus $I_3 \leq I_1$, $I_3 \leq I_2$ implies $I_3 \leq I_1 \wedge I_2$ … (5)

From (1), (4) and (5) we have any two elements I_1 and I_2 in $I(G)$ have a g.l.b $I_1 \wedge I_2$ in $I(G)$.

Hence $I(G)$ is a lattice.

Theorem 2.3: If $I(G)$, set of all *LI*–ideals of a commutative *l*–group implication algebra G then $I(G)$ is a distributive lattice.

Proof : From the previous theorem $I(G)$ is a lattice.

Then claim that $I_1 \vee (I_2 \wedge I_3) = (I_1 \vee I_2) \wedge (I_1 \vee I_3)$ for all I_1, I_2, I_3 in $I(G)$.

For let I_1, I_2, I_3 in $I(G)$ be arbitrary

Then we have $I_1 \vee (I_2 \wedge I_3) \leq (I_1 \vee I_2) \wedge (I_1 \vee I_3)$ …(1)

Let x in $(I_1 \vee I_2) \wedge (I_1 \vee I_3)$ be arbitrary

$\Rightarrow x$ in $I_1 \vee I_2$ and x in $I_1 \vee I_3 \Rightarrow x \leq a_1 \vee a_2$ and $x \leq a_3 \vee a_4$ where a_1, a_3 in I_1, a_2, a_4 in I_3.

$\Rightarrow x \leq a \vee a_2$ and $x \leq a \vee a_4$ where $a = a_1 \vee a_3 \Rightarrow x \leq (a \vee a_2) \wedge (a \vee a_4)$

$= a \vee [a_2 \wedge a_4]$ with a in I_1, $a_2 \wedge a_4$ in $I_2 \wedge I_3$

$\Rightarrow x$ in $I_1 \vee (I_2 \wedge I_3)$

Therefore $(I_1 \vee I_2) \wedge (I_1 \vee I_3) \leq I_1 \vee (I_2 \wedge I_3)$ …(2)

From (1) and (2) we get

$$I_1 \vee (I_2 \wedge I_3) = (I_1 \vee I_2) \wedge (I_1 \vee I_3) \text{ for all } I_1, I_2, I_3 \text{ in } I(G).$$

Hence $I(G)$ is a distributive lattice.

QUOTIENT COMMUTATIVE *L*-GROUP IMPLICATION ALGEBRA

In this section we introduced equivalence relation, equivalence class and commutative *l*-group implication algebra.

Definition 2.1 : Let I be an *LI*–ideal of a commutative *l*–group implication algebra G.

We define a binary reflection $\sim$ on G as follows:

$$x \sim y \Leftrightarrow (x \rightarrow y)' \in I \text{ and } (y \rightarrow x)' \in I \text{ for all } x, y \in G.$$

Then $\sim$ is an equivalence relation on G.

Definition 2.3 : We denote by I_x, the equivalence class containing x and by G/I the set of all equivalence classes of G with respect to "$\sim$". That is

$$I_x = \{y \in G / x \sim y\}$$

$$G/I = \{I_x / x \in G\}$$

Theorem 2.4 : If I is an *LI*–ideal of commutative *l*–group implication algebra G, then G/I is a commutative *l*–group implication algebra.

This commutative *l*-group implication algebra G/I is called quotient commutative *l*-group implication algebra.

Proof : Given G is a commutative *l*–group implication algebra, I, an *LI*–ideal of G and "$\sim$" is an equivalence relation defined above.

Denote $G/I = \{I_x / x \in G\}$.

Then it is easy to prove G/I is a commutative *l*-group implication algebra with respect to

(i) $I_x + I_y = I_{x+y}$ (ii) $I_x \vee I_y = I_{x \vee y}$

(iii) $I_x \wedge I_y = Ix \wedge y$ (iv) $I_x \rightarrow I_y = I_{x \rightarrow y}$

(v) $I_{x'} = I_x'$

with greatest element I_1 and least element I_0.

HOMOMORPHISM, ISOMORPHISM THEOREMS

In this section homomorphism, kernal of homomorphism and isomorphism are introduced and established fundamental theorem of homomorphism, first isomorphism theorem and second isomorphism theorem.

Definition 4.1: Let G_1, G_2 be two commutative *l*–group implication algebras. A map $\phi : G_1 \rightarrow G_2$ is called homomorphism if

(i) $\phi(a + b) = \phi(a) + \phi(b)$

(ii) $\phi(a \vee b) = \phi(a) \vee \phi(b)$

(iii) $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$

(iv) $\phi(a \rightarrow b) = \phi(a) \rightarrow \phi(b)$

(v) $\phi(a') = [\phi(a)]'$ for all $a, b \in G$.

Definition 4.2: Let $\phi : G_1 \rightarrow G_2$ be a homomorphism of commutative *l*–group implication algebra. Then the kernel of ϕ is defined by Ker $\phi = \{x \in G_1 / \phi(x) = 0\}$

Theorem 4.1: Let $\phi : G_1 \rightarrow G_2$ be a homomorphism of commutative *l*-group implication algebra. Then Ker ϕ is an *LI*–ideal of *G*.

Proof : Given $\phi : G_1 \rightarrow G_2$ is a homomorphism of commutative *l*-group implication algebra and Ker ϕ or $K_\phi = \{x \in G_1 / \phi(x) = 0 \text{ in } G_2\}$

To prove Ker ϕ is an *LI*-ideal

i.e., to prove (1) Ker $\phi \neq \phi$, Ker $\phi \leq G_1$

(2) $x, y \in$ Ker $\phi \Rightarrow x - y, x \vee y, x \wedge y \in$ Ker ϕ

(3) $0 < a < x, x \in$ Ker $\phi \Rightarrow a \in$ Ker ϕ

(4) $(x \rightarrow y) \in$ Ker ϕ, $y \in$ Ker $\phi \Rightarrow x \in$ Ker ϕ

For (1) :

We have $0 \in G_1$ such that $\phi(0) = 0$

$\Rightarrow 0 \in K_\phi \Rightarrow$ Ker $\phi \neq \phi$

Let $x \in K_\phi$ be arbitrary

$\Rightarrow x \in G_1$ such that $\phi(x) = 0 \Rightarrow x \in G_1$

Therefore Ker $\phi \subseteq G_1$

For (2) :

Let $x, y \in K_\phi$

$\Rightarrow x, y \in G_1$ such that $\phi(x) = 0, \phi(y) = 0$

$\Rightarrow x - y, x \vee y, x \wedge y \in G$, such that

$\phi (x - y) = \phi [x + (-y)] = \phi (x) + \phi (- y) = \phi (x) - \phi (y) = 0 - 0 = 0$

$\phi (x \vee y) = \phi (x) \vee \phi (y) = 0 \vee 0 = 0$

$\phi (x \wedge y) = \phi (x) \wedge \phi (y) = 0 \wedge 0 = 0$

$\Rightarrow \; x - y, x \vee y, x \wedge y \in \text{Ker } \phi$

For (3) :

Let $0 < a < x, \;\; x \in \text{Ker } \phi$

$\Rightarrow \;\; 0 < a < x, x \in G$, such that $\phi (x) = 0 \;\Rightarrow\; a \in G$, such that $\phi (a) < \phi (x) = 0$

$\Rightarrow \;\; a \in$ G, such that $\phi (a) = 0 \Rightarrow \;\; a \in \text{Ker } \phi$.

For (4) :

Let $(x \to y)' \in \text{Ker } \phi$, $y \in \text{Ker } \phi$

$\Rightarrow \;\; (x \to y)' \in G$, such that $\phi [(x \to y)'] = 0$

$y \in G$, such that $\phi (y) = 0$

$\Rightarrow \phi [(x \to y)'] = \phi [(x \to y)]' = [\phi (x) \to \phi (y)]' = [\phi (x) \to 0]' = \{[\phi (x)]'\}'$

$= \phi (x)$ with $x \in G$,

$\Rightarrow \; x \in \text{Ker } \phi$

Hence Ker ϕ is an *LI*–ideal of *G*.

Theorem 4.2: Let $\phi : G \to \{0, 1\}$ be an onto homomorphism of commutative *l*–group implication algebra. Then the kernel of ϕ, K_ϕ is a maximal *LI*–ideal of *G*.

Proof: Given $\phi : G \to \{0, 1\}$ is an onto homomorphism of commutative *l*-group implication algebra and $K_\phi = \{x \in G/\phi (x) = 0\}$

$\Rightarrow \;\; K_\phi \neq \phi$ since ϕ is onto $\Rightarrow \;\; K_\phi$ is an *LI*–ideal of *G*

To prove K_ϕ is a maximal *LI*–ideal.

Suppose K_ϕ is not maximal

Then there is a proper *LI*–ideal *I* containing K_ϕ

$\Rightarrow$ there exists $x, y \in G$ such that $x \in G - I, \;\; y \in I - K_\phi$

$\Rightarrow \; f(x) = f(y) = 1 \Rightarrow f(x \to y) = f(x) \to f(y) = 1 \to 1 = 1$

$\Rightarrow \; f[(x \to y)'] = [f(x \to y)]' = [f(x) \to f(y)]' = 1' = 0 \;\Rightarrow\;\; (x \to y)' \in K_\phi$

$\Rightarrow K_\phi \subset I, \; y \in I$

$\Rightarrow \;\; x \in I$ which is a contradiction.

Hence K_ϕ is a maximal *LI*–ideal of *G*.

Definition 4.3 : Let $\phi : G_1 \to G_2$ be a homomorphism of commutative *l*–group implication algebra. Then ϕ is called an isomorphism if ϕ is $1 - 1$ and onto. G_1 is isomorphic to G_2 if there exist an isomorphism.

Theorem 4.3 : (Fundamental theorem of homomorphism). Let *G* be a commutative *l*–group implication algebra and *I* an *LI*–ideal of *G*. Then the quotient commutative *l*–group implication algebra *G*/*I* is a homomorphic image of a commutative *l*–group implication algebra *G*. Conversely let G_1 and G_2 be commutative

l–group implication algebra and let $\phi : G_1 \to G_2$ be an onto homomorphism. Then G_1 / Ker ϕ is isomorphic to G_2.

Proof:

First part:

Given G is a commutative l-group implication algebra and I an LI-ideal of G.

Then G/I is a commutative l-group implication algebra.

To prove G/I is the homomorphic image of G.

Define : $f : G \to G/I$ by $f(x) = I_x$ for all x in G.

***f* is well-defined :**

$x = y \implies f(x) = f(y)$ where x, y in G

Suppose $x = y$

$\implies x - y = 0,\ \ 0 \in \ker f \implies x - y \in \ker f \implies f(x - y) = 0 \implies f(x) + f(-y) = 0$

$\implies f(x) - f(y) = 0 \implies f(x) = f(y)$

***f* is a homomorphism :**

Let x, y in G be arbitrary.

Then $f(x + y) = I_{x+y} = I_x + I_y = f(x) + f(y)$

$f(x \vee y) = I_{x \vee y} = I_x \vee I_y = f(x) \vee f(y)$

$f(x \wedge y) = I_{x \wedge y} = I_x \wedge I_y = f(x) \wedge f(y)$

$f(x \to y) = I_{x \to y} = I_x \to I_y = f(x) \to f(y)$

$f(x') = I_x' = (I_x)' = [f(x)]'$

***f* is onto:**

Take any element I_x in G/I

$\implies x$ in $G \implies f(x) = I_x$

Hence f is homomorphic to G to G/I.

Second part:

Given $\phi : G_1 \to G_2$ is an onto homomorphism of commutative l-group implication algebra and Ker $\phi = K = \{x \in G_1 / \phi(x) = 0\}$

Then K is a LI – ideal of G and G_1/K is a commutative l-group implication algebra

To prove G_1/K is isomorphic to G_2

Define $f : G_1/K \to G_2$ by $f(K_x) = \phi(x)$ where $x \in G_1$, $K_x \in G_1/k$

Then we claim that f is a well defined isomorphism of G_1/K onto G_2

***f* is well defined;** $K_x = K_y \implies f(K_x) = f(K_y)$, where $K_x, K_y \in G_1/k$

Suppose $K_x = K_y \implies x = y \implies \phi(x) = \phi(y) \implies f(K_x) = f(K_y)$

***f* is one-one:** $f(K_x) = f(K_y) \implies K_x = K_y$ where $K_x, K_y \in G_1/k$

Suppose $f(K_x) = f(K_y) \implies \phi(x) = \phi(y) \implies \phi(x) - \phi(y) = 0 \implies \phi(x) + \phi(-y) = 0$

$\implies \phi[x + (-y)] = 0 \implies \phi(x - y) = 0 \implies x - y = 0 \implies x = y \implies K_x = K_y$

***f* is onto:**

Take any $x_2 \in G_2$

$\Longrightarrow$ there exist $x_1 \in G_1$ such that $\phi(x_1) = x_2 \Longrightarrow Kx_1 \in G_1/K$

$$F(Kx_1) = \phi(x_1) = x_2$$

Thus for any $x_2 \in G_2$ there exist $K_{x1} \in G_1/K$ such that $f(K_{x1}) = x_2$

***f* preserves +, ∨, ∧, →, ′ :**

Let $K_a, K_b \in G_1/K$ be arbitrary $\Longrightarrow a, b \in G$

Then $f(K_a + K_b) = f(K_{a+b}) = \phi(a+b) = \phi(a) + \phi(b) = f(K_a) + f(K_b)$

$$f(K_a \vee K_b) = f(K_{a \vee b}) = \phi(a \vee b) = \phi(a) \vee \phi(b) = f(K_a) \vee f(K_b)$$

$$f(K_a \wedge K_b) = f(K_{a \wedge b}) = \phi(a \wedge b) = \phi(a) \wedge \phi(b) = f(K_a) \wedge f(K_b)$$

$$f(K_a \to K_b) = f(K_{a \to b}) = \phi(a \to b) = \phi(a) \to \phi(b) = f(K_a) \to f(K_b)$$

for all $K_a, K_b \in G_1/K$

Thus f is an onto isomorphism

$\Longrightarrow G_1/K$ isomorphic to G_2

Theorem 4.4 : Let G_1, G_2 and G_3 be commutative l–group implication algebras. $H : G_1 \to G_2$ an onto homomorphism and $g : G_1 \to G_3$ homomorphism with non-empty kernals.

If Ker $(h) \subseteq$ Ker (g) then there is a unique homomorphism $f : G_2 \to G_3$ satisfying $f \, o \, h = g$

Proof: Given $h : G_1 \to G_2$ is an onto homomorphism and $g : G_1 \to G_3$ a homomorphism with non-empty kernals of commutative l-group implication algebras.

Suppose Ker $(h) \subseteq$ Ker (g)

Then to prove (i) There is a unique homomorphism $f : G_2 \to G_3$

(ii) $f \, o \, h = g$

Take any $y \in G_2$

$\Longrightarrow$ there exist $x \in G_1$, such that $h(x) = y$

For the element $x \in G_1$, put $z = g(x) \in G_3$

Define $f : G_2 \to G_3$ by $f(y) = z$

Claim (1) f is well defined

(2) $f \, o \, h = g$

(3) f is a homomorphism

(4) f is unique.

For 1:

$y_1 = y_2 \Longrightarrow f(y_1) = f(y_2)$ where $y_1, y_2 \in G_2$

$y_1 = h(x_1)$, $y_2 = h(x_2)$ where $x_1, x_2 \in G_1$

Suppose $y_1 = y_2$

$\Longrightarrow h(x_1) = h(x_2) = y$ (say)

$y = h(x_1) = h(x_2)$ for $x_1, x_2 \in G_1$

$1 = h(x_1) \to h(x_2) = h(x_1 \to x_2)$

$0 = [h(x_1 \to x_2)]' = h(x_1 \to x_2)'$

$(x_1 \to x_2)' \in \text{Ker}(h) \subseteq \text{Ker}(g)$

$\Rightarrow \quad 0 = g[(x_1 \to x_2)'] = g(x_1 \to x_2)' = [g(x_1) \to g(x_2)]' \quad \Rightarrow \quad g(x_1) \to g(x_2) = 1$

$\Rightarrow \quad g(x_1) < g(x_2)$

Similarly, $g(x_2) < g(x_1)$

Therefore $g(x_1) = g(x_2)$

For 2 :

Let $x \in G_1$ be arbitrary, then

$(f \, o \, h)(x) = f[h(x)] = f[y] = z = g(x)$ for all $x \in G$

$\Rightarrow f \, o \, h = g$

For 3 : *f* is a homomorphism

Let $y_1, y_2 \in G_2$ be arbitrary

Then $f(y_1 \to y_2) = z_1 \to z_2 = f(y_1) \to f(y_2)$

$f(y_1 \vee y_2) = z_1 \vee z_2 = f(y_1) \vee f(y_2)$

$f(y_1 \wedge y_2) = z_1 \wedge z_2 = f(y_1) \wedge f(y_2)$

$f(y_1 + y_2) = z_1 + z_2 = f(y_1) + f(y_2)$

For 4 : *f* is unique:

Suppose there exist f_1, f_2 such that

$$f_1 \, o \, h = g, \quad f_2 \, o \, h = g$$

$\Rightarrow \quad f_1 \, o \, h = f_2 \, o \, h \Rightarrow (f_1 \, o \, h)(x) = (f_2 \, o \, h)(x), \quad x \in G$

$\Rightarrow \quad f_1[h(x)] = f_2[h(x)] \Rightarrow f_1[y] = f_2[y]$ for all y

$\Rightarrow \quad f_1 = f_2$

Therefore f is unique.

Theorem 4.5 : First Isomorphism theorem

Let $\phi : G \to G_1$ be an onto commutative l–group implication algebra homomorphism with $K_\phi = S$. If I_1 is an *LI*-ideal of G_1, then

$I = \phi^{-1}(I_1) = \{x \text{ in } G/\phi(x) \text{ in } I_1\}$ is an *LI*-ideal of G and S is a subset of I. Conversely if I is an *LI*-ideal of G containing S, then

$I_1 = \phi(I) = \{x_1 \text{ in } G_1/x_1 = \phi(x)$, for some x in $I\}$ is an *LI*-ideal of G_1 and $G/I \cong G_1/I_1$ Moreover, $G/I \cong (G/S)/(I/S)$.

Proof :

First part : Given $\phi : G \to G_1$ is an onto commutative l-group implication algebra homomorphism with $K_\phi = S$ and I_1 is an *LI*-ideal of G_1

To prove $I = \phi^{-1}(I_1) = \{x \text{ in } G/\phi(x) \text{ in } I_1\}$ is an *LI*-ideal of G and $S \subset I$.

Clearly, $I \neq \phi$, since $0 \in G$ such that $\phi(0) = 0$ in I_1

(i) Let $x. \, y$ in I be arbitrary

$\Rightarrow \; x, y$ in G such that $\phi(x), \phi(y)$ in I_1

$\Rightarrow$ $x + y, x \vee y, x \wedge y, x \to y, x'$ in G such that

$\phi(x + y) = \phi(x) + \phi(y)$ with $\phi(x) + \phi(y)$ in I_1

$\phi(x \vee y) = \phi(x) \vee \phi(y)$ with $\phi(x) \vee \phi(y)$ in I_1

$\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ with $\phi(x) \wedge \phi(y)$ in I_1

$\phi(x \to y) = \phi(x) \to \phi(y)$ with $\phi(x) \to \phi(y)$ in I_1

$\phi(x') = [\phi(x)]'$ with $[\phi(x)]'$ in I_1

$\Rightarrow$ $x + y, x \vee y, x \wedge y, x \to y, x'$ in G such that $\phi(x + y), \phi(x \vee y), \phi(x \wedge y), \phi(x \to y), \phi(x')$ in I_1

$\Rightarrow$ $x + y, x \vee y, x \wedge y, x \to y, x'$ in I

(ii) Let $0 < x < a$ and a in I

$\Rightarrow$ $0 < \phi(x) < \phi(a)$, $\phi(a)$ in I_1

$\Rightarrow$ $\phi(x)$ in I_1 $\Rightarrow$ x in I_1

(iii) Let x in S be arbitrary $\Rightarrow x \in K_\phi$

$\Rightarrow$ x in G such that $\phi(x) = 0$ in I_1

$\Rightarrow$ $\phi(x)$ in I_1 $\Rightarrow$ x in I

Therefore, $S \subset I$

Second Part:

Given $\phi : G \to G_1$ is an onto commutative *l*-group implication algebra homomorphism. I is an *LI*-ideal of G and

$$I_1 = \phi(I) = \{x_1 \text{ in } G_1 / x_1 = \phi(x), \text{ for some } x \text{ in } I\}$$

Clearly $I_1 \neq \phi$, since $0 \in G_1$, $0 = \phi(0)$

To prove:

I_1 is an *LI* – ideal of G_1.

(i) Let a_1, b_1 in I_1 be arbitrary

$\Rightarrow a_1, b_1$ in G_1 such that $a_1 = \phi(a)$, $b_1 = \phi(b)$ for some a, b in I

$\Rightarrow a_1 - b_1, a_1 \vee b_1, a_1 \wedge b_1, a_1 \to b_1, a_1'$ in G_1 such that

$a_1 - b_1 = \phi(a) - \phi(b) = \phi(a - b)$ with $\phi(a - b)$ in I_1, $a - b \in I$

$a_1 \vee b_1 = \phi(a) \vee \phi(b) = \phi(a \vee b)$ with $\phi(a \vee b)$ in I_1, $a \vee b \in I$

$a_1 \wedge b_1 = \phi(a) \wedge \phi(b) = \phi(a \wedge b)$ with $\phi(a \wedge b)$ in I_1, $a \wedge b \in I$

$a_1 \to b_1 = \phi(a) \to \phi(b) = \phi(a \to b)$ with $\phi(a \to b)$ in I_1, $a \to b \in I$

$a_1' = \phi(a') = [\phi(a)]'$ with $[\phi(a)]'$ in I_1, $a' \in I$

$\Rightarrow$ $a_1 - b_1, a_1 \vee b_1, a_1 \wedge b_1, a_1 \to b_1, a_1'$ in I_1

Let $0 < x_1 < a_1$ and a_1 in I_1

$\Rightarrow$ $0 < x < a$ and a in $I \Rightarrow$ x in $I \Rightarrow$ $x_1 = \phi(x)$, $\phi(x) \in I_1 \Rightarrow$ $x_1 \in I_1$

Therefore I_1 is an *LI*-ideal of G_1

Third Part:

To prove $G/I \cong G_1/I_1$

Define $g : G_1 \rightarrow G/I$ by $g(x_1) = I_x$, $x_1 = \phi(x)$

***g* is well defined:**

Suppose $I_x = I_y$

$\Rightarrow x = y \Rightarrow \phi(x) = \phi(y) \Rightarrow I_{\phi(x)} = I_{\phi(y)} \Rightarrow g(I_x) = g(I_y)$

Therefore g is well defined.

***g* is a homomorphism :**

Let $x_1, y_1 \in G_1$ be arbitrary.

$\Rightarrow x_1 = \phi(x)$, $y_1 = \phi(y)$ where $x, y \in G$.

Then, $g(x_1 + y_1) = I_{x+y} = I_x + I_y = g(x_1) + g(y_1)$

$g(x_1 \vee y_1) = I_{x \vee y} = I_x \vee I_y = g(x_1) \vee g(y_1)$

$g(x_1 \wedge y_1) = I_{x \wedge y} = I_x \wedge I_y = g(x_1) \wedge g(y_1)$

$g(x_1 \rightarrow y_1) = I_{x \rightarrow y} = I_x \rightarrow I_y = g(x_1) \rightarrow g(y_1)$

$g(x_1') = I_{x'} = (I_x)' = [g(x_1)]'$

Ker $g = I_1$:

Let $x_1 \in \ker g$ be arbitrary

$\Rightarrow x_1 \in G_1$ such that $g(x_1) = I_0 \Rightarrow x_1 = \phi(0)$, $\phi(0) \in I_1$

$\Rightarrow x_1 \in I_1$

Therefore Ker $g \subset I_1$

Conversely, let $x_1 \in I_1$ be arbitrary

$\Rightarrow x_1 = \phi(x)$, $x \in I$

$\Rightarrow x_1 = \phi(0)$, $0 \in I$

$\Rightarrow g(x_1) = I_0 \Rightarrow x_1 \in$ Ker g

Therefore $I_1 \subset$ Ker g

Hence $I_1 =$ Ker g

Hence we have g is an onto homomorphism with Ker $g = I_1$

$\Rightarrow G_1/I_1 \cong G/I$ by previous theorem

$\Rightarrow G/I \cong G_1/I_1$

Take $G_1 = G/S$ and $I_1 = I/S$

We get $G/I \cong (G/S)/(I/S)$

Theorem 4.6: Second isomorphism theorem

If I_1 and I_2 are two *LI*-ideals of a commutative *l*-group implication algebra G, then $I_1/I_1 \wedge I_2 \cong I_1 + I_2/I_2$

Proof : Given I_1 and I_2 are *LI*-ideals of G.

Then $I_1 + I_2$, $I_1 \wedge I_2$ are *LI*-ideals of G.

To prove $I_1/I_1 \wedge I_2 \cong I_1 + I_2/I_2$

Consider a map $\phi : I_1 \rightarrow I_1 + I_2/I_2$ by $\phi(a_1) = Ia_1$ where a_1 in I_1

Then we claim that ϕ is a well defined onto homomorphism with Ker $\phi = I_1 \wedge I_2$

ϕ is well defined:

Suppose $a_1 = a_2$ where a_1, a_2 in I_1

$\Rightarrow$ $a_1 - a_2 = 0$ in I_1

$\Rightarrow$ $Ia_1 = Ia_2 \Rightarrow \phi(a_1) = \phi(a_2)$

Therefore ϕ is well defined

ϕ is onto :

Take any I_a in $I_1 + I_2 / I_2$

$\Rightarrow$ a in $I_1 + I_2$ $\Rightarrow$ $a = a_1 + a_2$ for some a_1 in I_1, a_2 in I_2

$\Rightarrow$ $I_a = I(a_1 + a_2) = Ia_1 + Ia_2 = Ia_1$

$\phi(a) = Ia_1$

Therefore ϕ is onto.

ϕ is a homomorphism :

Let a_1, a_2 in I_1 be arbitrary.

Then, $\phi(a_1 + a_2) = I_{(a1 + a2)} = I_{a1} + I_{a2} = \phi(a_1) + \phi(a_2)$

$\phi(a_1 \vee a_2) = I_{(a1 \vee a2)} = I_{a1} \vee I_{a2} = \phi(a_1) \vee \phi(a_2)$

$\phi(a_1 \wedge a_2) = I_{(a1 \wedge a2)} = I_{a1} \wedge I_{a2} = \phi(a_1) \wedge \phi(a_2)$

$\phi(a_1 \rightarrow a_2) = I_{(a1 \rightarrow a2)} = I_{a1} \rightarrow I_{a2} = \phi(a_1) \rightarrow \phi(a_2)$

$\phi(a_1') = I(a_1') = (Ia_1)' = [\phi(a_1)]'$ for all a_1, a_2 in I_1

Therefore ϕ is a homomorphism

Ker ϕ = $I_1 \cap I_2$:

Let a in $I_1 \cap I_2$ be arbitrary

$\Rightarrow$ a in I_1 and a in I_2 $\Rightarrow$ a in I_1 and $I_0 = I_2$ $\Rightarrow$ a in I_1 such that $\phi(a) = I_2$

$\Rightarrow$ a in ker ϕ

Therefore, $I_1 \wedge I_2 \subseteq$ Ker ϕ …(1)

Conversely, let a_1 in Ker ϕ be arbitrary

$\Rightarrow$ a_1 in I_1 such that $\phi(a_1)$ = zero element in $I_1 + I_2/I_2$

$\Rightarrow$ a_1 in I_1 such that $\phi(a_1) = I_0 = I_2$ $\Rightarrow$ a_1 in I_1 and a_1 in $I_2 = I_0$ $\Rightarrow$ a_1 in $I_1 \wedge I_2$

Therefore, Ker $\phi \subseteq I_1 \wedge I_2$ ...(2)

From (1) and (2) we get, Ker $\phi = I_1 \wedge I_2$

Hence $\phi : I_1 \rightarrow I_1 + I_2/I_2$ is an onto homomorphism with Ker $\phi = I_1 \wedge I_2$.

$\Rightarrow$ $I_1/I_1 \wedge I_2 \cong I_1 + I_2/I_2$ by previous theorem.

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