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DISTRIBUTIVE *LI*-IDEAL IN COMMUTATIVE *l*-GROUP IMPLICATION ALGEBRA

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In this paper, to introduce distributive *LI*-ideal, Characterization theorem for distributive *LI*-ideal, dually distributive *LI*-ideal, Characterization theorem for dually distributive *LI*-ideal and the relation between them in a commutative *l*-group implication algebra *G*.

KEY WORDS: *l*-group, commutative *l*-group, *LI*-ideal, distributive *LI*-ideal, dually distributive *LI*-ideal.

INTRODUCTION

It is well known that a distributed complimented lattice is a Boolean algebra which is equivalent to Boolean ring with identity. This relation gives a link between Lattice theory and Modern Algbra. The algebraic structure connecting lattice and group is called *l*-group or lattice ordered group. Many common abstractions, namely Dually residuated lattice ordered semi groups, lattice ordered commutative groups, lattice ordered near rings lattice ordered semi rings and commutative *l*-group implication algebra are presented in [8], [4], [1], [7] and [5] respectively. The concept of *LI*-ideal in lattice implication algebra is introduced in [9].

Ore, O., has introduced and developed the concept of distributive element in a lattice. The concept of distributive ideal is called distributive element in the ideal lattice *I* (*L*) of a lattice *L* has been introduced by Gratzer, G., and Schmidt, E.T.,

In this paper the concept of distributive *LI*-ideal, dually distributed *LI*-ideal are introduced and established it characteristic theorems.

PRELIMINARIES

In this section are listed a number of definitions and results which are made use of throughout the paper. The symbols $\leq$, $+$, $-$, $\vee$, $\wedge$, $\rightarrow$, $*$ and $\in$ will denote inclusion, sum, difference, join (least upper bound), meet (greatest lower bound), implication, symmetric difference and membership in a lattice *L* or commutative *l*-group implication algebra *G*. Small letters *a, b*,…. will denote elements of the lattice *L* or commutative *l*-group *G*.

Definition 1.1: A non-empty set *G* is called an *l*-group iff

(i) $(G, +)$ is a group (ii) $(G, \leq)$ is a lattice

(iii) If $x \leq y$, then $a + x + b \leq a + y + b$, for all a, b, x, y in G.

Or

$(a + x + b) \vee \vee (a + y + b) = (a + x\vee \ \vee y + b)$

$(a + x + b) \wedge \wedge (a + y + b) = (a + x \wedge \wedge y + b)$, for all a, b, x, y in G.

Definition 1.2 : An l-group G is called commutative l-group if $x + y = y + x$ for all x, y in G.

Definition 1.3 : An implication algebra is a non-empty set L with greatest element I, least element 0, an unary operation " ' " and a binary operation "→" which satisfies the following axioms:

(I1) $1 \to x = x$ (I2) $x \to x = \mathrm{I}$ (I3) $(x \to y) \to y = (y \to x) \to x$

(I4) $(((y \to z) \to z) \to x) \to x = (((y \to x) \to x) \to z) \to z$

(I5) $x \to (y \to z) = y \to (x \to z)$ (I6) $0 \to x = \mathrm{I}$

(I7) $x \to 0 = x'$ for all $x, y, z \in L$.

Definition 1.4 : Let $(L, \vee \vee, \wedge \wedge, 0, I)$ be a bounded lattice with an order-reversing involution $'$, I and 0 the greatest and the smallest element of L respectively, $\to: L \times L \to L$ be a mapping. Then $(L, \vee \vee, \wedge \wedge, ', \to, \mathrm{O}, \mathrm{I})$ is called a lattice implication algebra if the following conditions hold for any $x, y, z \in L$:

(L_1) $x \to (y \to z) = y \to (x \to z)$, (L_2) $x \to x = \mathrm{I}$, (L_3) $x \to y = y' \to x'$,

(L_4) If $x \to y = y \to x = \mathrm{I}$, then $x = y$, (L_5) $(x \to y) \to y = (y \to x) \to x$,

(L_6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$ (L_7) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

The binary operation "→" will be denoted by juxt a position. We can define a partial ordering "≤" on a lattice implication algebra L by $x \le y$ if and only if $x \to y = 1$.

Theorem 1.1: Definitions 1.3 and 1.4 are equivalent.

Theorem 1.2. In a lattice implication algebra L, the following are hold

(i) $x \le y$ if and only if $x \to y = 1$ (ii) $x \le (x \to y) \to y$

(iii) $0 \to x = 1$, $1 \to x = x$ and $x \to 1 = 1$ (iv) $x' = x \to 0$

(v) $x \to y \le (y \to z)\ (x \to z)$ (vi) $(x \vee y) = (x \to y) \to y$

(vii) $x \le y \Longrightarrow y \to z \le x \to z$ and $z \to x \le z \to y$.

Definition 1.5 : A non-empty set G is called **commutative *l*-group implication algebra** if only if

1. $(G, +)$ is a commutative group 2. $(G, \to)$ is an implication algebra
3. $x \le \mathrm{y} \Longrightarrow$ (i) $a + x \le a + y$

   (ii) $(a \to x) \to b \ge (a \to y) \to b$

   (iii) $a \to (x \to b) \ge a \to (y \to b)$, for all a, b, x, y in G.

Definition 1.6 : A non empty set G is called **commutative *l*-group implication algebra** if and only if

1. $(G, +)$ is a commutative group
2. $(G, \to)$ is an implication algebra
3. (i) $a + (x \vee y) = (a + x) \vee (a + y)$

   (ii) $a + (x \wedge y) = (a + x) \wedge (a + y)$

(iii) $[a \to (x \vee y)] \to b] = [(a \to x) \to b] \wedge [(a \to y) \to b]$

$= a \to [(x \vee y) \to b]$

(iv) $[a \to (x \wedge y)] \to b] = [(a \to x) \to b] \vee [(a \to y) \to b]$

$= a \to [(x \wedge y) \to b]$, for all x, y, a, b in G.

Theorem 1.3 : The above two definitions for commutative l-group implication algebra are equivalent.

Definition 1.7 : Let G be a commutative l-group implication algebra and I a non-empty subset of G. Then I is called an LI-ideal if and only if

1. a, b in I implies $a - b$ in I
2. a, b in I implies $a \vee b$, $a \wedge b$ in I
3. $0 < x < a$, and a in I implies x in I
4. $(x \to y)' \in I$ and $y \in I$ imply $x \in I$

In a commutative l – group implication algebra, $\{0\}$, G are LI – ideals of G.

Theorem 1.4 : If I_1, I_2, are two LI-ideals of commutative l -group implication algebra G, then

(i) $I_1 \vee I_2 = \{x \in G / x \leq x_1 \vee x_2$ for some x_1 in I_1, x_2 in $I_2\}$ is an LI-ideal

(ii) $I_1 \wedge I_2 = \{x \in G / x$ in I_1 and x in $I_2\}$ is an LI – ideal

(iii) $I_1 + I_2 = \{x$ in $G / x \leq x_1 + x_2$ for some x_1 in I_1, x_2 in $I_2\}$ is an LI-ideal

(iv) $I_1 \vee I_2$ is the smallest LI-ideal containing $I_1 \cup I_2$

Theorem 1.5 : Let G be a commutative l-group implication algebra and $I(G)$, set of all LI-ideals of G. Then $I(G)$ is a lattice.

DISTRIBUTIVE *LI*-IDEAL

In this section distributive LI-ideal is introduced and established characterization theorem for distributive LI-ideal.

Definition 2.1: An LI–ideal D of a commutative l-group implication algebra G is called a distributive LI – ideal if $D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y)$ for all $X, Y \in I(G)$

Example 2.1 : Every ideal of a Boolean algebra B is called a distributive LI-ideal.

Proof : Given D is an ideal of a Boolean algebra B.

$\Rightarrow$ (i) $a, b \in D \Rightarrow a \vee b \in D$ (ii) $a < b, b \in I \Rightarrow a \in I$

To prove that B is a distributive LI-ideal

That is to prove

(i) a, b in $D \Rightarrow a - b$ in D

(ii) a, b in $D \Rightarrow a \vee b$, $a \wedge \vee \wedge b \in D$

(iii) $0 < x < a$, and a in $D \Rightarrow x$ in D

(iv) $(x \to y)' \in D$ and $y \in D \Rightarrow x \in D$

(v) $D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y)$ for all $X, Y \in I(B)$

For (i):

Let $a, b \in D$

$\Rightarrow$ $a, b \in D$, $a \geq a - b$

$\Rightarrow$ $a - b \in D$, by the definition of ideal.

For (ii) :

Let $a, b \in D$ $\Rightarrow$ $a, b \in D$, $a \wedge b \leq a$

$\Rightarrow$ $a \wedge b \in D$, by the definition of ideal

Also $a, b \in D$ $\Rightarrow$ $a \vee b \in D$, by the definition of ideal

For (iii) :

Let $a \in I$, $0 < x < a \Rightarrow x \in D$, by the definition of ideal

For (iv) :

Given $(x \rightarrow y)' \in D$ and $y \in D \Rightarrow$ $y \vee (x \rightarrow y)' \in D$ …(1)

To prove $x \in D$

Consider $y \vee (x \rightarrow y)' = (y \rightarrow (x \rightarrow y)') \rightarrow (x \rightarrow y)' = ((x \rightarrow y) \rightarrow y') \rightarrow (x \rightarrow y)'$

$$= (x \rightarrow y) \rightarrow (y')' = (x \rightarrow y) \rightarrow y = x \vee y$$

$\Rightarrow$ $x \vee y \in D$, by (1)

$\Rightarrow$ $x \in D$, since $x < x \vee y$

Hence D is a *LI*-ideal.

For (v):

Given B is a Boolean algebra

$\Rightarrow$ B is a commutative *l*-group implication algebra

We know that $I(B)$, the set of all *LI*-ideals of a commutative *l*-group implication algebra form a distributive lattice.

$\Rightarrow$ $D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y)$, for all $X, Y \in I(B)$ and $D \in I(B)$

$\Rightarrow$ D is a distributive *LI*-ideal.

Example 2.2 : Every *LI*-ideal of a commutative *l*-group implication algebra is a distributive *LI*-ideal.

Theorem 2.1: Charactererization theorem for distributive *LI* – ideal

Let D be an *LI*– ideal of a commutative *l*–group implication algebra G. Then the following conditions are equivalent.

(i) D is distributive

(ii) The map $\phi : X \rightarrow D \vee X$ is an onto homomorphism of $I(G)$ onto

$$[D) = \{X \text{ in } I(G) / X \geq D\}$$

(iii) The binary relation θ_D on $I(G)$ is defined by

"$X \equiv Y (\theta_D) \Leftrightarrow D \vee X = D \vee Y$ where X, Y in $I(G)$"

is a congruence relation.

Proof : Let $X, Y, Z \in I(G)$ be arbitrary.

(i) $\Rightarrow$ (ii):

ϕ preserves $\vee$:

Then

$\phi (X \vee Y) = D \vee (X \vee Y) = (D \vee D) \vee (X \vee Y) = D \vee [D \vee (X \vee Y)]$

$= [D \vee (D \vee X)] \vee Y = [(D \vee X) \vee D] \vee Y = [(D \vee X) \vee (D \vee Y)]$

$= \phi (X) \vee \phi (Y)$

Thus $\phi (X \vee Y) = \phi (X) \vee \phi (Y)$, for all $X, Y \in I(G)$.

ϕ preserves $\wedge$:

Then $\phi (X \wedge Y) = D \vee (X \wedge Y) = [(D \vee X) \wedge (D \vee Y)]$, by (i)

$= \phi (X) \wedge \phi (Y)$

Thus $\phi (X \wedge Y) = \phi (X) \wedge \phi (Y)$, for all $X, Y \in I(G)$.

ϕ is onto :

Take any X in $[D)$

$\Rightarrow$ X in $I(G)$ such that $X \geq D \Rightarrow X$ in $I(G)$ such that $D \vee X = X \Rightarrow \phi (X) = D \vee X = X$

Thus for any X is $[D)$ there exist $X \in I(G)$ such that $\phi (X) = X$.

Hence ϕ is an on to homomorphism.

(ii) $\Rightarrow$ (iii) :

We claim that

1) θ_D is reflexive. 2) θ_D is symmetric. 3) θ_D is transitive. 4) substitution property $X \equiv X_1 (\theta_D)$, $Y \equiv Y_1 (\theta_D)$

$\Rightarrow$ $X \vee Y \equiv X_1 \vee Y_1 (\theta_D)$

$\Rightarrow$ $X \wedge Y \equiv X_1 \wedge Y_1 (\theta_D)$, for all X, X_1, Y, Y_1 in $I(G)$

For (1) :

Then $D \vee X = D \vee X$

$\Rightarrow$ $X \equiv X (\theta_D)$

Thus $X \equiv X (\theta_D)$, for all $X \in I(G)$

For (2) :

Suppose $X \equiv Y (\theta_D) \Rightarrow D \vee X = D \vee Y \Rightarrow D \vee Y = D \vee X \Rightarrow Y \equiv X (\theta_D)$

Thus $X \equiv Y (\theta_D) \Rightarrow Y \equiv X (\theta_D)$, for all $X, Y \in I(G)$.

For (3) :

Suppose $X \equiv Y (\theta_D)$ and $Y \equiv Z (\theta_D)$

$\Rightarrow$ $D \vee X = D \vee Y$ and $D \vee Y = D \vee Z \Rightarrow D \vee X = D \vee Z \Rightarrow X \equiv Z (\theta_D)$

Thus $X \equiv Y (\theta_D)$ and $Y \equiv Z (\theta_D)$ implies $X \equiv Z (\theta_D)$, for all $X, Y, Z \in I(G)$.

For (4) :

Let $X, X_1, Y, Y_1 \in I(G)$ be arbitrary.

Suppose $X \equiv X_1(\theta_D)$, $Y \equiv Y_1(\theta_D) \Rightarrow D \vee X = D \vee X_1$, $D \vee Y = D \vee Y_1$

Now $D \vee (X \vee Y) = (D \vee X) \vee Y = (D \vee X_1) \vee Y = (X_1 \vee D) \vee Y = X_1 \vee (D \vee Y)$

$= X_1 \vee (D \vee Y_1) = (X_1 \vee D) \vee Y_1 = (D \vee X_1) \vee Y_1 = D \vee (X_1 \vee Y_1)$

Similarly $D \vee (X \wedge Y) = D \vee (X_1 \wedge Y_1) \Rightarrow X \vee Y = (X_1 \vee Y_1) (\theta_D)$

and $X \wedge Y = (X_1 \wedge Y_1) (\theta_D)$

Thus $X \equiv X_1\ (\theta_D)$ and $Y \equiv Y_1(\theta_D)$ implies

$X \vee Y = (X_1 \vee Y_1)(\theta_D)$ and $X \wedge Y = (X_1 \wedge Y_1)\ (\theta_D)$, for all $X, X_1, Y, Y_1 \in I(G)$.

Hence θ_D is a congruence relation.

(iii) ⟹ (i):

Claim: $D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y)$ for all $X, Y \in I(G)$.

Then $D \vee X = (D \vee D) \vee X = D \vee (D \vee X)$

$D \vee Y = (D \vee D) \vee Y = D \vee (D \vee Y)$

$\Rightarrow$ $X \equiv (D \vee X)\ (\theta_D)$, $Y \equiv (D \vee Y)\ (\theta_D) \Rightarrow X \wedge Y = (D \vee X) \wedge (D \vee Y)\ (\theta_D)$, by (iii)

$\Rightarrow$ $D \vee (X \wedge Y) = D \vee [(D \vee X) \wedge (D \vee Y)]$, by the definition of θ_D

$\Rightarrow$ $D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y)$, since $D \le (D \vee X) \wedge (D \vee Y)$

Thus $D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y)$ for all $X, Y \in I(G)$.

Hence D is a distributive *LI*-ideal.

DUALLY DISTRIBUTIVE LI-IDEAL

In this section dually distributive *LI*-ideal is introduced and established characterization theorem for dually distributive *LI*-ideal.

Definition 3.1: An *LI*-ideal D of a commutative l–group implication algebra G is called dually distributive *LI*-ideal if $D \wedge (X \vee Y) = (D \wedge X) \vee (D \wedge Y)$ for all $X, Y \in I(G)$.

Example 3.1 : Every ideal of Boolean algebra B is a dually distributive *LI*-ideal.

Proof: Every ideal D of a Boolean algebra B is a distributive *LI*-ideal.

$\Rightarrow D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y)$ for all $D, X, Y \in I(G)$, .. .(1)

To prove $D \wedge (X \vee Y) = (D \wedge X) \vee (D \wedge Y)$ for all $X, Y \in I(G)$

Now $(D \wedge X) \vee (D \wedge Y) = [(D \wedge X) \vee D] \wedge [(D \wedge X) \vee Y]$, by (1)

$= [D \vee (D \wedge X)] \wedge [Y \vee (D \wedge X)] = D \wedge [(Y \vee D) \wedge (Y \vee X)]$, by (1)

$= [D \wedge [(D \vee Y)] \wedge (Y \vee X)] = D \wedge (X \vee Y)$, for all $X, Y \in I(G)$

$\Rightarrow D$ is a dually distributive *LI*-ideal.

Example 3.2 : Every *LI*-ideal of a commutative l-group implication algebra G is a dually distributive *LI*-ideal.

Theorem 3.1: Charactererization theorem for dually distributive *LI*-ideal

Let D be a *LI*-ideal of a commutative l-group implication algebra G. Then the following conditions are equivalent.

(i) D is dually distributive.

(ii) The map $\phi : X \to D \wedge X$ is a homomorphism of

$I(G)$ onto $(D] = \{X \text{ in } I(G)/X \le D\}$

(iii) The binary relation θ_D on $I(G)$ is defined by

"$X \equiv Y\ (\theta_D) \Leftrightarrow D \wedge X = D \wedge Y$ where X, Y in $I(G)$"

is a congruence relation.

Proof: Follows dually.

Theorem 3.2 : If *D* is an *LI*-ideal in a commutative *l*-group implication algebra *G*, then the following are equivalent.

(1) *D* is a distributive *LI*-ideal

(2) *D* is a dually distributive *LI*-ideal.

Proof: Follows from the following results

(1) *I* (*G*) is distributive lattice

(2) Every distributive lattice is dually distributive.

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