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A PARTIAL NON-NEGATIVE P_0-MATRIX COMPLETION PROBLEM FOR 6 × 6 MATRICES WITH 6 VERTICES AND 4 ARCS

DR. A. RAMESH KUMAR

Head, Deptt. of Mathematics, Srimad Andavan Arts and Science College (Autonomous), T.V. Kovil, Trichy-5

AND

M.S. PONMUDI

Asst. Prof. Deptt. of Mathematics, Srimad Andavan Arts and Science College (Autonomous), T.V. Kovil, Trichy-5

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We Study on a Partial Non-negative P_0-matrix completion problem is considered 6 × 6 matrices specifying digraphs for $p = 6$, $q = 4$, where p is number of vertices and q is number of arcs by performing zero completion on the matrices. The study establishes that all digraphs for $p = 6$, $q = 4$ specifying 6 × 6 partial matrices which are either cycles or acyclic digraphs have non-negative P_0-completion.

KEYWORDS : P_0-matrix, Nonnegative P_0-matrix, Principal sub matrix, partial matrix, matrix completion.

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INTRODUCTION

A Partial matrix is a matrix in which some entries are specified while the remaining unspecified entries are free to be chosen. For example, $A = \begin{bmatrix} 4 & 2 & x \\ 2 & 1 & y \\ 4 & -1 & 1 \end{bmatrix}$ a 3 × 3 partial matrix with elements in positions (1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2) (3, 3) specified while elements in positions (1, 3), (2, 3) are unspecified. A fully specified principal submatrix such as A (1, 2) of matrix A above has all entries specified. A completion of a partial matrix is a particular choice of values for unspecified so that the resulting matrix specifies a certain property. An $n \times n$ matrix has a list of positions given by $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$. If Q is a subset of this list of positions, then Q is said to be pattern of $n \times n$ matrix. A partial matrix specifies the pattern if its specified entries are those exactly listed in the pattern. For example, the partial matrix A above specifies the pattern {(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2) (3, 3)}. Matrices are of various classes such as positive definite, P, P_0, M_0, nonnegative P_0 matrices and others. Each of the class specifies certain properties. As states in [1], for a particular class Π of matrices, a pattern is said to have Π-completion if every partial Π-matrix specifying the pattern can be completed to a Π-matrix. If there exists even one partial Π-matrix specifying the pattern that cannot be completed that pattern is said not to have completion. Digraphs assist the study of nonnegative P_0-matrix completion since the case considered involve patterns involving 6 × 6 matrices with all diagonal entries specified and not necessarily for position $\{j, i\}$ to be in the pattern if position $\{i, j\}$ is in the pattern.

Definition 1.1 : A real $n \times n$ matrix called a P_0-matrix if all its principal minors are non-negative. A partial P_0-matrix is a partial matrix in which all fully specified sub-matrices are P_0-matrices.

Definition 1.2 : A real $n \times n$ matrix is nonnegative P_0-matrix if all entries are non-negative and all its principal minors are non-negative. *i.e.*, it's a P_0-matrix whose entries are nonnegative. A partial matrix is a partial non-negative P_0-matrix if determinants of all fully specified sub-matrices are non-negative and all specified entries are non-negative.

Definition 1.3 : A pattern is said to have a nonnegative P_0-matrix specifying the pattern can be completed to a non-negative P_0-matrix.

Definition 1.4 : A digraph is ordered pair $D = (V, A)$ comprising of a set of vertices together with a set A of directed edges called arcs. The order of a digraph is the number of vertices in the digraph while the size of a digraph is the number of arcs in the digraph.

Definition 1.5 : A digraph H is said to be a sub-digraph of D if every vertex of H is also a vertex of D and every arc of H is also an arc of D.

Definition 1.6 : Let D be a digraph, a path that begins and ends at the same vertex is called a cycle. A digraph that does not contain any cycles is called an acyclic digraph.

Definition 1.7 : A chord is an arc joining two non-consecutive vertices of a cycle. A digraph is chordal if any cycle of length > 3 has a chord. A subset of a directed graph is called a clique if it contains atleast three vertices and for each pair of vertices v_i and v_j in the subset, both $v_i \to \mathrm{v_j}$ and $v_i \to v_j$ are true.

In many situations it is convenient to permute entries of a partial matrix. A permutation matrix P is obtained by interchanging rows on the identity matrix. The permutation matrix A is $PA\ P^T$. This is represented on the digraph by renumbering the vertices. As a result of the following lemma we are allowed to permute a partial non-negative P_0-matrix and hence renumber digraph vertices as convenient.

CLASSIFICATION OF 6 × 6 MATRICES WITH 6 VERTICES AND 4 ARCS

Lemma 2.1 [1] :

The class of non-negative P_0 – matrices is closed under permutation.

Some studies have been done on non-negative P_0–matrix completion. In [4], Hogben established that for non-negative P_0-matrices, patterns of every non-separable strongly connected induced sub-digraph has non-negative P_0-completion. In the same study it is shown that all 3 × 3 matrices have non-negative P_0-completion prove of which is given in [2]. In [5], Hogben established that a pattern that has non-negative P_0-completion also have non-negative P-completion. In [2], it is established that a 4 × 4 matrix that includes all diagonal positions has non-negative P_0-completion if and only if its digraph is complete when it has a 4-cycle. Also shown in the study is that any positionally symmetric pattern that includes all diagonal positions and whose graph is an n-cycle has non-negative P_0-completion if and only if $n \neq 4$.

In this section all possible digraphs with 6 vertices and 4 arcs are considered and 6 × 6 partial matrices specifying the digraphs extracted. The construction of digraphs will be with the guidance of graphs with six points and four lines as given in [3].

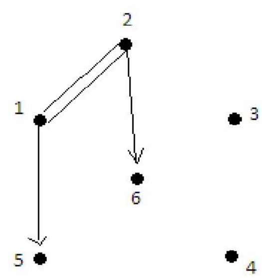
The process of extracting the partial non-negative P_0-matrices will be as follows : A specific entry a_{ij} will be used to represent the corresponding present arc in the digraph, an unspecified entry x_{ij} will represent a corresponding missing arc in the digraph while d_{ii} will specify the diagonal entries. Zero completion method will then be used to find out whether each of the cases have zero completion to non-negative P_0-matrix or not.

Application of the digraph

3.1 Case (i) :

Six vertices and four arcs in acyclic graphs (any two vertices of two digraphs):

Consider the digraph below:



Let $A = \begin{pmatrix} d_{11} & a_{12} & x_{13} & x_{14} & a_{15} & x_{16} \\ a_{21} & d_{22} & x_{23} & x_{24} & x_{25} & a_{26} \\ x_{31} & x_{32} & d_{33} & x_{34} & x_{35} & x_{36} \\ x_{41} & x_{42} & x_{43} & d_{44} & x_{45} & x_{46} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} & x_{56} \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & d_{66} \end{pmatrix}$ be a partial non-negative P_0-matrix representing the digraph above.

Determining the determinants of all the principal minors then setting the unspecified entries to zero.

$x_{13} = 0, x_{14} = 0, x_{16} = 0, x_{23} = 0, x_{24} = 0, x_{25} = 0, x_{31} = 0, x_{32} = 0, x_{34} = 0, x_{35} = 0, x_{36} = 0,$
$x_{41} = 0, x_{42} = 0, x_{43} = 0, x_{45} = 0, x_{46} = 0, x_{51} = 0, x_{52} = 0, x_{53} = 0, x_{54} = 0, x_{56} = 0, x_{61} = 0,$
$x_{62} = 0, x_{63} = 0, x_{64} = 0, x_{65} = 0.$

Determinant of the principal sub matrices will be as follows:

Det. A (1, 2) = $d_{11}d_{22} - a_{12}a_{21} \geq 0$. Since A (1, 2) is fully specified.

Similarly, Det A (1, 3), Det A (1, 4), Det A (1, 5), Det A (1, 6), Det A (2, 3), Det A (2, 4), Det A (2, 5), Det A (2, 6), Det A (3, 4), Det A (3, 5), Det A (3, 6), Det A (4, 5), Det A (4, 6), Det A (5, 6) ≥ 0.

Det A (1, 2, 3) = $d_{11}d_{22}d_{33} - a_{12}a_{21}d_{33} = d_{33}\,(d_{11}d_{22} - a_{12}a_{21}) \geq 0$. Since A (1, 2) is fully specified.

Similarly, Det A (1, 2, 4), Det A (1, 2, 5), Det A (1, 2, 6), Det A (1, 3, 4), Det A (1, 3, 5), Det A (1, 3, 6), Det A (1, 4, 5), Det A (1, 4, 6), Det A (2, 3, 4), Det A (2, 3, 5), Det A (2, 3, 6), Det A (2, 4, 5), Det A (2, 4, 6), Det A (3, 4, 5), Det A (3, 4, 6), Det A (4, 5, 6) ≥ 0.

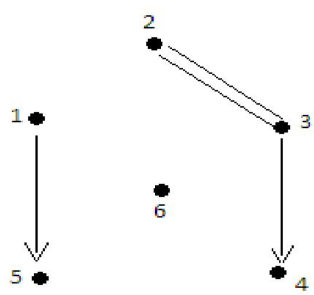
Det A (1, 2, 3, 4) $= d_{11}d_{22}d_{33}d_{44} - a_{12}d_{11}d_{33}d_{44} = d_{11}d_{33}d_{44}\,(d_{22} - a_{12}) \geq 0$.

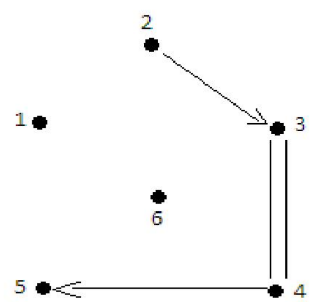
Similarly, Det A (1, 2, 3, 5), Det A (1, 2, 3, 6), Det A (1, 3, 4, 5), Det A (1, 3, 4, 6), Det A (1, 3, 5, 6), Det A (1, 2, 4, 6), Det A (1, 2, 5, 6), Det A (2, 3, 4, 5), Det A (2, 3, 4, 6), Det A (3, 4, 5, 6) ≥ 0.

Det $A = d_{11}d_{22}d_{33}d_{44}d_{55}d_{66} - a_{12}a_{21}d_{33}d_{44}d_{55}d_{66} = d_{33}d_{44}d_{55}d_{66}\,(d_{11}d_{22} - a_{12}a_{21}) \geq 0$.

Hence all principal minors are non-negative and therefore partial matrix has zero completion into non-negative P_0-matrix.

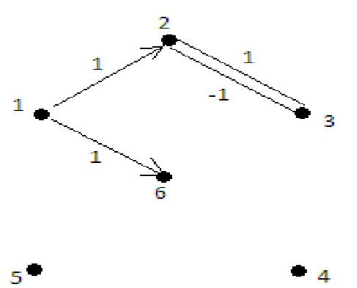
Carrying out similar procedures for the following digraph's similar results will be obtained.





Example: 3.1.1 (i)

Consider the digraph below



Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$ be a partial non negative P_0-matrix representing the digraph above. Determining the determinants of all the principal minors then setting the unspecified entries to zero.

Determinant of the principal sub matrices will be as follows:

Det A (1, 2) = $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ = 2 ≥ 0. Similarly, Det A (1, 3), Det A (1, 4), Det A (1, 5), Det (1, 6), Det A (2, 3), Det A (2, 4), Det A (2, 5), Det A (2, 6), Det A (3, 4), Det A (3, 5), Det A (3, 6), Det A (4, 5), Det A (3, 6), Det A (5, 6) ≥ 0.

Det A (1, 2, 3) = $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 3 \end{pmatrix}$ = 7 ≥ 0. Similarly, Det A (1, 2, 4), Det A (1, 2, 5), Det A (1, 2, 6), Det A (1, 3, 4), Det A (1, 3, 5), Det A (1, 3, 6), Det A (1, 4, 5), Det A (1, 4, 6), Det A (2, 3, 4), Det A (2, 3, 5), Det A (2, 3, 6), Det A (2, 4, 5), Det A (2, 4, 6), Det A (3, 4, 5), Det A (3, 4, 6), Det A (4, 5, 6) ≥ 0.

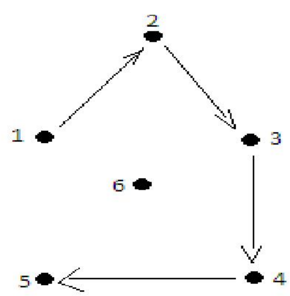
Det A (1, 2, 3, 4) = $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ = 108 ≥ 0. Similarly, Det A (1, 2, 3, 5), Det A (1, 2, 3, 6), Det A (1, 3, 4, 5), Det A (1, 3, 4, 6), Det A (1, 3, 5, 6), Det A (1, 2, 4, 6), Det A (1, 2, 5, 6), Det A (2, 3, 4, 5), Det A (2, 3, 4, 6), Det A (3, 4, 5, 6) ≥ 0.

Hence all principal minors are non-negative and therefore partial matrix has zero completion into non-negative P_0-matrix.

3.2 Case (ii)

Six vertices having four arcs in acyclic graph (one digraph):

Consider the digraph below:



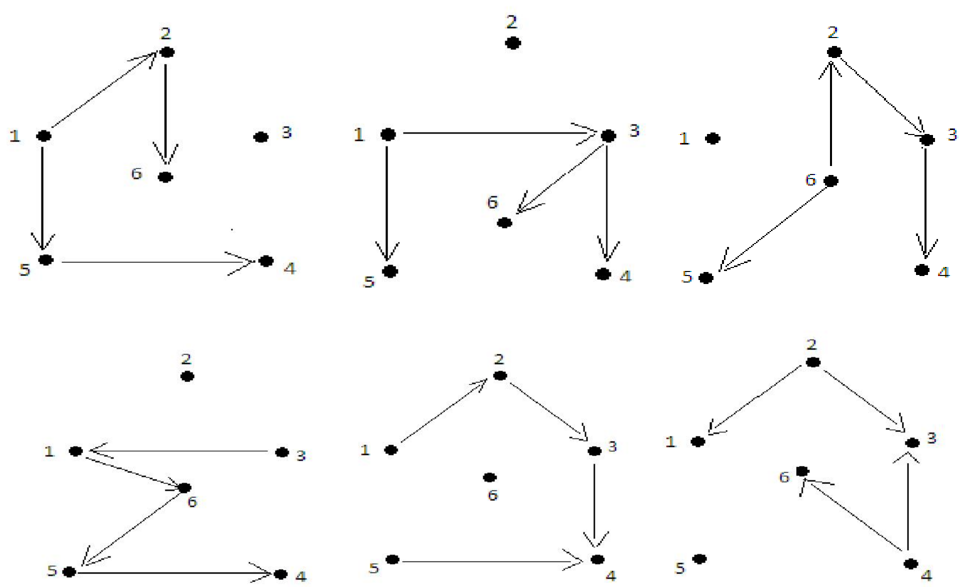
Let $A = \begin{pmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} & x_{36} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} & x_{46} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} & x_{56} \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & d_{66} \end{pmatrix}$ be a partial non negative P_0-matrix representing the digraph above. Determining the determinants of all the principal sub matrices, the unspecified entries to zero.

$x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$, $x_{16} = 0$, $x_{21} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{26} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{35} = 0$, $x_{36} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{46} = 0$, $x_{51} = 0$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$, $x_{56} = 0$, $x_{61} = 0$, $x_{62} = 0$, $x_{63} = 0$, $x_{64} = 0$, $x_{65} = 0$.

Determinants of the principal sub matrices will be follows: Det A (1, 2), Det A (1, 3), Det A (1, 4), Det A (1, 5), Det A (1, 6), Det A (2, 3), Det A (2, 4), Det A (2, 5), Det A (2, 6), Det A (3, 4), Det A (3, 5), Det A (3, 6), Det A (4, 5), Det A (3, 6), Det A (5, 6), Det A (1, 2, 3), Det A (1, 2, 4), Det A (1, 2, 5), Det A (1, 2, 6), Det A (1, 3, 4), Det A (1, 3, 5), Det A (1, 3, 6), Det A (1, 4, 5), Det A (1, 4, 6), Det A (2, 3, 4), Det A (2, 3, 5), Det A (2, 3, 6), Det A (2, 4, 5), Det A (2, 4, 6), Det A (3, 4, 5), Det A (3, 4, 6), Det A (4, 5, 6), Det A (1, 2, 3, 4), Det A (1, 2, 3, 5), Det A (1, 2, 3, 6), Det A (1, 3, 4, 5), Det A (1, 3, 4, 6), Det A (1, 3, 5, 6), Det A (1, 2, 4, 6), Det A (1, 2, 5, 6), Det A (2, 3, 4, 5), Det A (2, 3, 4, 6), Det A (3, 4, 5, 6) ≥ 0.

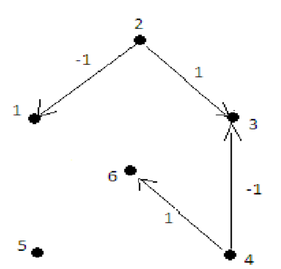
Hence all principal minors are non negative and therefore partial matrix has zero completion into non negative P_0-matrix.

Carrying out similar procedures to all other acyclic digraphs shown below, similar results will be obtained.



Example: 3.2.1(ii)

Consider the digraph below



Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$ be a partial non negative P_0-matrix representing the digraph above. Determining the determinants of all the principal minors then setting the unspecified entries to zero.

Determinant of the principal sub matrices will be as follows:

Det A (1, 2) = $\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = 2 \geq 0$. Similarly, Det A (1, 3), Det A (1, 4), Det A (1, 5), Det A (1, 6), Det A (2, 3), Det A (2, 4), Det A (2, 5), Det A (2, 6), Det A (3, 4), Det A (3, 5), Det A (3, 6), Det A (4, 5), Det A (3, 6), Det A (5, 6) ≥ 0.

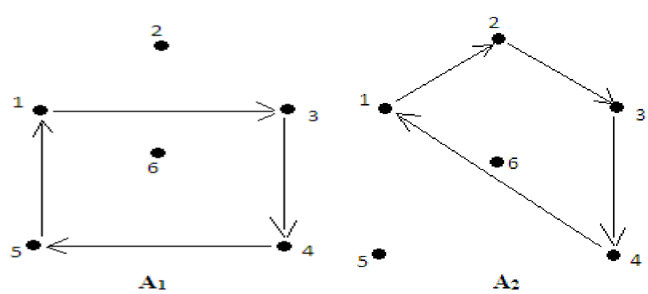
Det A (2, 3, 4) = $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & -1 & 4 \end{pmatrix} = 24 \geq 0$. Similarly, Det A (1, 2, 3), Det A (1, 2, 4), Det A (1, 2, 5), Det A (1, 2, 6), Det A (1, 3, 4), Det A (1, 3, 5), Det A (1, 3, 6), Det A (1, 4, 5), Det A (1, 4, 6), Det A (2, 3, 5), Det A (2, 3, 6), Det A (2, 4, 5), Det A (2, 4, 6), Det A (3, 4, 5), Det A (3, 4, 6), Det A (4, 5, 6) ≥ 0.

Det A (1, 2, 4, 6) = $\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 6 \end{pmatrix} = 192 \geq 0$. Similarly, Det A (1, 2, 3, 4), Det A (1, 2, 3, 5), Det A (1, 2, 3, 6), Det A (1, 3, 4, 5), Det A (1, 3, 4, 6), Det A (1, 3, 5, 6), Det A (1, 2, 5, 6), Det A (2, 3, 4, 5), Det A (2, 3, 4, 6), Det A (3, 4, 5, 6) ≥ 0.

Hence all principal minors are non-negative and therefore partial matrix has zero completion into non-negative P_0-matrix.

3.3 Case (iii)

Cyclic graph: Consider the following digraph which is a cycle



Let

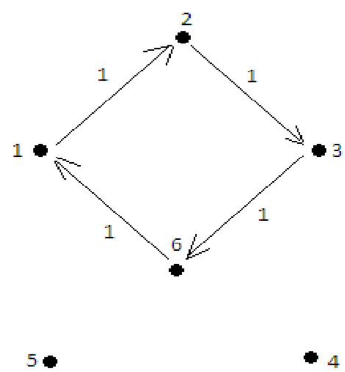
$$A_1 = \begin{pmatrix} d_{11} & x_{12} & a_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & d_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} & x_{36} \\ x_{41} & x_{42} & x_{43} & d_{44} & a_{45} & x_{46} \\ a_{51} & x_{52} & x_{53} & x_{54} & d_{55} & x_{56} \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & d_{66} \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} d_{11} & a_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & d_{22} & a_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & d_{33} & a_{34} & x_{35} & x_{36} \\ a_{41} & x_{42} & x_{43} & d_{44} & x_{45} & x_{46} \\ x_{51} & x_{52} & x_{53} & x_{54} & d_{55} & x_{56} \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & d_{66} \end{pmatrix}$$

be a partial non negative P_0-mattrix representing the digraph above. Determining the deteminants of all the principal minors then setting the unspecified entries to zero. Determinants the principal sub-matrices can be shown as above to be ≥ 0.

Hence all principal minors are non negative and therefore partial matrix has zero completion into non-negative P_0-matrix.

Example: 3.3.1 (iii)

Consider the digraph below



Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$ be a partial non-negative P_0-matrix representing the digraph above. Determining the determinants of all the principal minors then setting the unspecified entries to zero.

Determinant of the principal sub matrices will be as follows:

Det A (3,4) $= \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = 12 \geq 0$. Similarly, Det A (1, 2), Det A (1, 3), Det A (1, 4), Det A (1, 5), Det A (1, 6), Det A (2, 3), Det A (2, 4), Det A (2, 5), Det A (2, 6), Det A (3, 5), Det A (3, 6), Det A (4, 5), Det A (3, 6), Det A (5, 6) ≥ 0.

$$\text{Det } A\ (3, 4, 5) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} = 60 \geq 0.$$ Similarly, Det A(1, 2, 3), Det A (1, 2, 4), Det A (1, 2, 5), Det A (1, 2, 6), Det A (1, 3, 4), Det A (1, 3, 5), Det A (1, 3, 6), Det A (1, 4, 5), Det A (1, 4, 6), Det A (2, 3, 4), Det A (2, 3, 5), Det A (2, 3, 6), Det A (2, 4, 5), Det A (2, 4, 6), Det A (3, 4, 5), Det A (3, 4, 6), Det A (4, 5, 6) ≥ 0.

$$\text{Det } A\ (1, 3, 4, 6) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 6 \end{pmatrix} = 288 \geq 0.$$ Similarly, Det A (1, 2, 3, 4), Det A (1, 2, 3, 5), Det A (1, 2, 3, 6), Det A (1, 3, 4, 5), Det A (1, 2, 4, 6), Det A (1, 3, 5, 6), Det A (1, 2, 5, 6), Det A (2, 3, 4, 5), Det A (2, 3, 4, 6), Det A (3, 4, 5, 6) ≥ 0.

Hence all principal minors are non-negative and therefore partial matrix has zero completion into non-negative P_0-matrix.

Conclusion

Hence we conclude that, the graphs and digraphs have been used effectively to study matrix completion problems. For positionally symmetric pattern that includes all diagonal positions, the graph of Q/pattern graphs is used to carry out the study. For patterns without positional symmetry, digraph/directed graphs are used in matrix completion problems for pairs of related classes of matrices. Finally, all the digraphs for 6 × 6 matrices with 4 arcs which are either cycles or acyclic digraphs have zero completion into non-negative P_0-matrix.

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