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LINE-BLOCK GRAPHS AND CROSSING NUMBERS

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The graph valued function namely the *line-block graph* $L_b(G)$ of a graph G is the graph whose point set is the set of lines and blocks of G and two points are adjacent if the corresponding blocks contain a common cutpoint of G or one corresponds to a block B of G and other to a line e of G and e is in B. In this paper, we establish a necessary and sufficient condition for graphs whose line-block graphs have crossing number k, $k = 1, 2, 3$ or 4.

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INTRODUCTION

Throughout the paper, we only consider simple finite graphs. We follow [4, 5] for the terminology and notation. A *block* of a graph is a connected nontrivial graph having no cutpoint. The *block graph* $B(G)$ of a graph G is the graph whose points are the blocks of G and in which two points are adjacent whenever the corresponding blocks have a cutpoint in common. A graph G is called a *planar graph* if G can be drawn in a plane so that no two of its lines are cross each other. A graph that is not planar is called *nonplanar*. The *crossing number* $Cr(G)$ of a graph G is the minimum number of pairwise intersections of its lines when G is drawn in the plane. Obviously $Cr(G) = 0$ if and only if G is planar.

The *line-block graph* $L_b(G)$ of a graph G is the graph whose point set is the set of lines and blocks of G and two points are adjacent if the corresponding blocks contain a common cutpoint of G or one corresponds to a block B of G and other to a line e of G and e is in B. This concept was introduced by V. R. Kulli [6]. The crossing number of graph valued functions were studied in [1-3, 8-12].

The following theorem will be useful in the proof of our results.

Theorem 1.1[7] The line-block graph $L_b(G)$ of a graph G is planar if and only if every cutpoint of G is incident with at most 4 blocks.

LEMMAS

We start with preliminary lemmas, which are useful to prove our main theorems.

Lemma 2.1 If a graph G has a cutpoint incident with six blocks, then $\mathrm{Cr}(L_b(G)) \geq 3$.

Proof. Suppose G has a cutpoint c incident with six blocks. Then the points corresponding to the blocks incident with cutpoint c constitutes an induced subgraph K_6 in $L_b(G)$. It is known that $Cr(K_6) = 3$. Thus $Cr(L_b(G)) = 3$.

Suppose G has a cutpoint incident with at most five blocks. Suppose c_1 is a cutpoint incident with five blocks. Then the points corresponding to the blocks incident with cutpoint c_1 constitutes an induced subgraph K_5 in $L_b(G)$. It is known that $Cr(K_5) = 1$. It follows that $Cr(L_b(G)) \geq 3$.

Lemma 2.2 If a graph G has two cutpoints each of which is incident with six blocks, then $\mathrm{Cr}(L_b(G)) \geq 6$.

Proof. Suppose G has two cutpoints c_i, $i = 1, 2$ each of which is incident with six blocks. As in Lemma 2.1, points corresponding to six blocks incident with cutpoint c_i forms an induced subgraph K_6 which has crossing number 3. Since G has two specified cutpoints c_1 and c_2. So $Cr(L_b(G)) = 6$.

Suppose G has a cutpoint incident with at most five blocks. Suppose c_3 is a cutpoint incident with five blocks. Then the points corresponding to five blocks incident with cutpoint c_3 forms an induced subgraph K_5 in $L_b(G)$. It is known that $Cr(K_5) = 1$. It follows that $Cr(L_b(G)) \geq 6$.

Lemma 2.3 If G has k $(k \geq 1)$ cutpoints each of which is incident with five blocks and every other cutpoint of G is incident with at most four blocks, then $\mathrm{Cr}(L_b(G)) = k$.

Proof. Suppose G has k $(k \geq 1)$ cutpoints each of which is incident with five blocks. Then the points corresponding to five blocks incident with cutpoint c_i; $1 \leq i \leq k$ forms an induced subgraph isomorphic to K_5. It is known that $Cr(K_5) = 1$. Since G has k specified cutpoints c_i; $1 \leq i \leq k$, so $Cr(L_b(G)) = k$ and a cutpoint of G incident with at most four blocks forms K_2 or K_3 or K_4 as an induced subgraph, which is planar.

Main results

In the following theorem we deduce a necessary and sufficient condition for line-block graphs with crossing number 1 or 2.

Theorem 3.1. A graph G has a line-block graph with crossing number k; $k = 1$ or 2 if and only if G has exactly k cutpoints each of which is incident with five blocks and every other cutpoint of G is incident with at most four blocks.

Proof. Suppose $Cr(L_b(G)) = k$; $k = 1$ or 2. Then $L_b(G)$ is nonplanar. By Theorem 1.1, G has at least one cutpoint incident with at least five blocks. Assume G has a cutpoint incident with six blocks. Then by Lemma 2.1, $Cr(L_b(G)) \geq 3$, which is a contradiction. It implies that every cutpoint of G is incident with at most five blocks and G has at least one cutpoint incident with exactly five blocks. Suppose $c_1, c_2, \ldots, c_r$ be the cutpoints incident with five blocks in G. Then the points corresponding to five blocks incident with cutpoint c_i; $1 \leq i \leq r$ forms an induced subgraph isomorphic to K_5 in $L_b(G)$. Since $Cr(K_5) = 1$, and $L_b(G)$ has at least r (line-disjoint) copies of K_5, $L_b(G)$ has at least r crossings. If $r > k$, then $Cr(L_b(G)) \geq k$, a contradiction. This proves that $r \leq k$. We consider two cases depending on the value of k. Assume $k = 1$. Then $r = k$, for otherwise by Theorem 1.1, $Cr(L_b(G)) = 0$, a contradiction. Assume k =2. Suppose $r \leq 1$. Then $Cr(L_b(G)) \neq k$, a contradiction. Therefore $r = k$. Thus G has exactly k cutpoints each of which is incident with five blocks and every other cutpoint of G is incident with at most four blocks, since a cutpoint of G incident with at most four blocks forms K_2 or K_3 or K_4 as an induced subgraph, which is planar.

Conversely, suppose G has exactly k ($k = 1$ or 2) cutpoints of which is incident with five blocks and every other cutpoint of G is incident with at most four blocks. Then by Lemma 2.3, $Cr(L_b(G)) = k$, $k = 1$ or 2.

We now characterize the line -block graphs with crossing number 3.

Theorem 3.2 A graph G has a line-brock graph with crossing number 3 if and only if (i) or (ii) holds:

(i) G has exactly three cutpoints each of which is incident with five blocks and every other cutpoint of G is incident with at most four blocks.

(ii) G has a unique cutpoint incident with six blocks and every other cutpoint of G is incident with at most four blocks.

Proof. Suppose $Cr(L_b(G)) = 3$. Then clearly, $L_b(G)$ is nonplanar and by Theorem 1.1, G has at least one cutpoint incident with at least five blocks. We consider two distinct cases:

Case 1. Assume G has k ($k \neq 3$) cutpoints each of which is incident with five blocks and every other cutpoint of G is incident with at most four blocks. We consider two distinct subcases:

Subcase 1.1. Assume G has k ($k \leq 2$) cutpoints. Then by Theorem 3.1 , $Cr(L_b(G)) \leq 2$, which is a contradiction.

Subcase 1.2. Assume G has k ($k \geq 4$) cutpoints. Then by Lemma 2.3, $Cr(L_b(G)) \geq 4$, again a contradiction. Thus from these two subcases we conclude that G holds (i).

Case 2. Assume G has two cutpoints each of which is incident with six blocks. Then by Lemma 2.2, $Cr(L_b(G)) \geq 6$, which is a contradiction. Thus G holds (ii).

Conversely, suppose G is a graph satisfying (i) or (ii). Then by Theorem 1.1, $L_b(G)$ has at least one crossing. We now show that its crossing number is 3. First assume (i) holds. Then by Lemma 2.3, $Cr(L_b(G)) = 3$. Now assume (ii) holds. Then by Lemma 2.1, $Cr(L_b(G)) \geq 3$. But a cutpoint of G incident with at most four blocks forms K_2 or K_3 or K_4 as an induced subgraph, which is planar. It follows that $Cr(L_b(G)) = 3$.

We now establish a necessary and sufficient condition for graphs whose line-block graphs having crossing number 4.

Theorem 3.3 A graph G has a line-block graph with crossing number 4 if and only if (i) or (ii) holds:

(i) G has exactly four cutpoints each of which is incident with five blocks and every other cutpoint of G is incident with at most four blocks.

G has a unique cutpoint incident with five blocks and six blocks respectively and every other cutpoint of G is incident with at most four blocks.

Proof. Suppose $Cr(L_b(G)) = 4$. Then clearly, $L_b(G)$ is nonplanar and by Theorem 1.1, G has at least one cutpoint incident with at least five blocks. We consider the following two cases :

Case 1. Assume G has k ($k \neq 4$) cutpoints each of which is incident with five blocks and every other cutpoint of G is incident with at most four blocks. We consider the following two subcases:

Subcase 1.1. Assume G has k $(k \leq 3)$ cutpoints. Then by Theorems 3.1 and 3.2, $Cr(L_b(G)) \leq 3$, which is a contradiction.

Subcase 1.2. Assume G has k $(k \geq 5)$ cutpoints. Then by Lemma 2.3, $Cr(L_b(G)) \geq 5$, again a contradiction. Thus from these two subcases we conclude that G holds (i).

Case 2. In this case, we consider the following two subcases:

Subcase 2.1. Assume G has two cutpoints each of which is incident with five blocks and a unique cutpoint incident with six blocks such that every other cutpoint of G is incident with at most four blocks. Let c_1 and c_2 be the cutpoints incident with five blocks and let c_3 be the cutpoint incident with six blocks. Then by Theorem 3.1, the points corresponding to the blocks incident with cutpoints c_1 and c_2 constitute 2 crossings and by Lemma 2.1, the points corresponding to the blocks incident with cutpoint c_3 constitute at least 3 crossings in $L_b(G)$. Thus $L_b(G)$ has at least 5 crossings, which is a contradiction.

Subcase 2.2. Suppose G has a unique cutpoint incident with five blocks and two cutpoints each of which is incident with six blocks such that every other cutpoint of G is incident with at most four blocks. Let c_1 be the cutpoint incident with five blocks and let c_2 and c_3 be the cutpoints incident with six blocks. Then by Theorem 3.1, the points corresponding to the blocks incident with cutpoint c_1 forms 1 crossing and by Lemma 2.2, the points corresponding to the blocks incident with cutpoints c_2 and c_3 constitute at least 6 crossings in $L_b(G)$. Thus $L_b(G)$ has at least 7 crossings, again a contradiction. Thus from these two subcases we conclude that G holds (ii).

Conversely, suppose G is a graph satisfying (i) or (ii), then by Theorem 3.2, $L_b(G)$ has more than 3 crossings. We now show that it has exactly 4 crossings. First assume (i) holds, then by Lemma 2.3, $Cr(L_b(G)) = 4$. Now assume (ii) holds. Let c_1 and c_2 be the cutpoints incident with five and six blocks respectively. By Theorem 3.1, the points corresponding to the blocks incident with cutpoint c_1 constitute one crossing in $L_b(G)$. Since G has only one cutpoint incident with five blocks, By Lemma 2.1, the points corresponding to the blocks incident with cutpoint c_2 constitutes 3 crossing and a cutpoint of G incident with at most four blocks forms K_2 or K_3 or K_4 as an induced subgraph, which is planar. Thus $L_b(G)$ has exactly 4 crossings.

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References

1. Basavanagoud, B., Quasi-total graphs with crossing numbers, *J. Discrete Mathematical Sciences and Cryptography*, **1 (2-3)**, 133-142 (1998).
2. Basavanagoud, B., Veeragoudar, Jaishri B., The block-transformation graph G^{110} with crossing number one, *J. Computer and Mathematical Sciences*, **6(4)**, 222-227 (2015).
3. Guy, R. K., Latest results on crossing numbers in recent trends in graph theory, *Springer*, New York, 143-156 (1971).
4. Harary, F., *Graph Theory*, Addison-Wesley, Reading, Mass (1969).
5. Kulli, V. R., *College Graph Theory*, Vishwa International Publications, Gulbarga, India (2012).

6. Kulli, V. R., On line-block graphs, *International Research Journal of Pure Algebra*, **5(4)**, 40-44 (2015).
7. Kulli, V. R., Planarity of line-block graphs, *J. Computer and Mathematical Sciences*, **6(4)**, 206-209 (2015).
8. Kulli, V. R., Muddebihal, M. H., Total-block graphs and semitotal-block graphs with crossing numbers, *Far East J. Appl. Math.*, **4(1)**, 99-106 (2000).
9. Niranjan, K. M., Nagaraja, P., Lokesh, V., Semi-image neighborhood block graphs with crossing numbers, *J. Sci. Res.*, **5 (2)**, 295-299 (2013).
10. Patil, H. P., Forbidden subgraphs and total graphs with crossing number 1, *J. Mathematical and Physical Sci.,* **17**, 293-295 (1983).
11. Patil, H. P., Rengarasu, U., On iterated line graphs with crossing number one, *Indian J. Pure Appl. Math*., **25(10)**, 1059-1065 (1994).
12. Patil, H. P., Kulli, V. R., Middle graphs and crossing numbers, *J. Discussiones Mathematicae*, **7**, 97-106 (1985).

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