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ROW GENERALIZED ORTHOGONAL COMBINATORIAL MATRIX AND CERTAIN COMBINATORIAL DESIGNS

MITHILESH KUMAR SINGH

Professor, University Department of Mathematics, Ranchi University, Ranchi, India

AND

DINESH PRASAD

Research Scholar, University Department of Mathematic, Ranchi University, Ranchi, India

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In this paper we give some properties of row generalized orthogonal combinatorial matrix and some construction theorems on BIBDs, GD, RD and (r, λ) design using row generalized orthogonal combinatorial matrix.

KEYWORDS : Generalized orthogonal combinatorial matrix (GOCM), balanced incomplete block design (BIBD), group divisible design (GD), rectangular design (RD), (r, λ) design.

Introduction

Construction theory of combinatorial design has extensive interactions with different areas *viz.*, group theory, graph theory, number theory, finite field, finite geometry, and linear algebra. Bose [1], Shrikhande [11] constructed BIBDs using theory of groups and finite fields. Sane [10] constructed a family of symmetric design from afine plane. Bridges [3], Trang [14], Brourer and Wilbrink [4], Bose and Conor [2], Rao [9], Hanani [6], Zhu [16] constructed symmetric BIBDs and related designs. Deza *et al* [5] and Vanstone [15] contributed in the construction of (r, λ) designs.

In this paper we use the concept of GOCM [13] in the construction of BIBDs, GD, RD and (r, λ) designs. Generalized orthogonal matrix (GOM) over group algebra was introduced by Singh *et. al.* [12] in the construction of complex Hadamard matrix.

We give the following definition

Let $\mathbb{N}$ be the set of natural numbers,

1.0. Left *m*-module. Let R be the ring of $m \times m$ integer matrices, *i.e*. matrices with entries in $\mathbb{Z}$, then the set M of all $m \times n$ matrices, $m, n \in \mathbb{N}$ is a left R-module. M will be called **left *m*-module.**

1.1. Row Generalized Orthogonal Matrix (Row GOM) over left modules on the ring of $m \times m$ matrices over $\mathbb{Z}$.

We consider left modules M_j of $m \times n_j$ matrices on the ring of $m \times m$ matrices over $\mathbb{Z}$, ($j = 1, 2, ..., s$) such modules are left m-modules.

Let $N = (N_{ij})$, $i = 1, 2, 3$ ………, l and $j = 1, 2, 3,$ ……., s be a block matrix, where N_{ij} are $m \times n_j$ matrix from M_j, $j = 1, 2, \ldots, s$ where $\sum_{j=1}^{s} n_j = n$, clearly blocks of j^{th} column of $N \in M_j$.

Let $R_i = (N_{i1}\ N_{i2} \ldots\ldots\ N_{is})$ be the i^{th} block row of N.

We define inner product of two block rows R_i and R_j

as $R_i o R_j = R_i R_j^T = R_{ij} = \sum_{k=1}^{l} N_{ik} N_{jk}^T \in M_j$

N is called a row GOM if there exist fixed integers $r, \lambda_1, \lambda_2, \lambda_3$ such that

$$R_{ij} = \sum_{k=1}^{l} N_{ik} N_{jk}^T = rI_m + \lambda_1 K_m, \text{ whenever } i = j$$
$$\text{and } \lambda_2 I_m + \lambda_3 K_m \text{ whenever } i \neq j$$

$l, s, m, n_1, n_2, \ldots, n_s, r, \lambda_1, \lambda_2, \lambda_3$, will be called parameters of the row GOM.

If $n_1 = n_2 = \ldots = n_s = n$, then $l, s, m, n, r, \lambda_1, \lambda_2, \lambda_3$, will be called parameters of the row GOM.

When $m = n$, row GOM will has square blocks with parameters $l, s, n, r, \lambda_1, \lambda_2, \lambda_3$.

REMARK: A row GOM will be called a row GOCM if N_{ij} are (0, 1) $m \times n_j$ matrix from M_j.

1.2. Balanced incomplete block design (BIBD). A block design $D = (V, \beta)$, where V is a set of v elements called points and β is a collection of b subsets of V called blocks such that for some fixed r, k, λ

(i) Each block contains exactly k points

(ii) Each point belongs to exactly r blocks

(iii) Each pair of points occurs in exactly λ blocks,

is called a BIBD . Such a design is also called 2-(v, b, r, k, λ)-design or a (v, k, λ)-design.

1.3. (r, λ) **- Design**: A (r, λ) design is a block design (V, β) such that

(i) Every element of V occurs in precisely r blocks .

(ii) Every pair of distinct elements of V occurs in precisely λ blocks.

1.4. Association Scheme. A d-class association scheme with vertex set X of order v is a sequence of non zero {0, 1}-matrices $A_0, A_1, A_2, \ldots, A_d$ with rows and column indexed by X, such that

(i) $A_0 = I$,

(ii) $A_i^T = A_i$ for all $i \in \{0, 1, 2, \ldots\ldots, d\}$

(iii) $A_0 + A_1 + A_2 + \ldots\ldots + A_d = J$,

(iv) $A_i A_j$ lies in the real span of $A_0, A_1, A_2, \ldots, A_d$: $A_i A_j = \sum_{k=0}^{d} P_{ij}^{\ k} A_k$.

(vide Godsil and Song [7])

Properties of GOCM

(i) The row sum of a GOCM is r, which is independent of the row.

(ii) The column sum of a GOCM is k, which is independent of the column.

Remark : A square GOCM is called regular if its row sum is equal to its column sum = k, which is independent of a row or column.

2.1. TYPES OF GOCM

In this section we classify row GOCMs according to the representation of inner products of rows of a GOCM.

(1) Type I GOCM A row GOCM will be called type I row GOCM

if $\lambda_1 = \lambda_2 = \lambda_3$

(2) Type II GOCM A row GOCM will be called type II row GOCM

if $\lambda_1 = \lambda_3$ or $\lambda_2 = \lambda_3$

(3) Type III GOCM A row GOCM will be called type III row GOCM

if $\lambda_1 = \lambda_2$

(4) Type IV GOCM A row GOCM will be called type IV row GOCM

if it is not of type I, II or III.

Theorem 1. A row GOCM N with constant column sum k is in general a rectangular design (RD).

Proof: We have

$$NN^T = \begin{pmatrix} rI_m + \lambda_1 K_m & \dots & \lambda_2 I_m + \lambda_3 K_m \\ \vdots & \ddots & \vdots \\ \lambda_2 I_m + \lambda_3 K_m & \cdots & rI_m + \lambda_1 K_m \end{pmatrix}$$

$$= r\,(I_m \times I_l) + \lambda_1\,(I_m \times K_l) + \lambda_2\,(K_m \times I_l) + \lambda_3\,(K_m \times K_l).$$

Let, $B_0 = (I_m \times I_l)$, $B_1 = (I_m \times K_l)$, $B_2 = (K_m \times I_l)$, $B_3 = (K_m \times K_l)$.

Claim. These are the association matrices of at most three class association scheme:

Using the properties of Kronecker product it is easy to verify the postulates of AS

We have

(i) $B_0^T = B_0$,

$B_1^T = (I_m \times K_l)^T = I_m \times K_l = B_1$,

$B_2^T = (K_m \times I_l)^T = K_m \times I_l = B_2$,

$B_3^T = (K_m \times K_l)^T = K_m \times K_l = B_3$.

Hence, B_0, B_1, B_2, B_3 are symmetric matrices.

(ii) $B_0 + B_1 + B_2 + B_3 = J_m \times J_l$.

(iii) $B_1B_2 = (I_m \times K_l)\,(K_m \times I_l)$

$= K_m \times K_l = B_3$.

$B_1B_3 = (I_m \times K_l)\,(K_m \times K_l)$

$= (l-1)\,B_3 + (l-2)\,B_2$.

$B_2B_3 = (K_m \times I_l)\,(K_m \times K_l)$

$= (m-1)\,(I_m \times K_l) + (m-2)\,(K_m \times K_l)$

$= (m-1)\,B_1 + (m-2)\,B_3$.

$B_1^2 = (l-1)\,B_0 + (l-2)\,B_2$,

$B_2^2 = (m-1)\,B_0 + (m-2)\,B_2$,

$$B_3^2 = (m-1)(l-1) B_0 + (m-2)(l-2) B_3.$$

The above products give the values of p_{jk}^i; $(0 \le i, j, k \le 3)$ which are the parameters of a rectangular association scheme. Hence the theorem.

Theorem 2: A row GOCM N with constant column reduces to GD design when $\lambda_1 = \lambda_3$ or $\lambda_2 = \lambda_3$.

Proof : We proceed to show that when $\lambda_2 = \lambda_3$, N is the incidence matrix of a GD design. We have

$$NN^T = r(I_m \times I_l) + \lambda_1 (I_m \times K_l) + \lambda_2 \{(K_m \times I_l) + (K_m \times K_l)\}$$
$$= r(I_m \times I_l) + \lambda_1 (I_m \times K_l) + \lambda_2 (K_m \times J_l).$$

Let $B_0 = I_m \times I_l,\ B_1 = I_m \times K_l,\ B_2 = K_m \times J_l.$

$$B_1^2 = (l-1) B_0 + (l-2) B_1,$$

$\therefore$ $n_1 = p_{11}^0 = l - 1.$

$$B_2^2 = l(m-1) B_0 + l(m-1) B_1 + l(m-2) B_2,$$

$\therefore$ $n_2 = p_{22}^0 = l(m-1).$

$$B_1 B_2 = (l-1) B_2.$$

The above products give the values of p_{jk}^i; $(0 \le i, j, k \le 2)$ which are the parameters of a GD association scheme.

Now, let $\lambda_1 = \lambda_3$ then

$$NN^T = r(I_m \times I_l) + \lambda_1 (K_m \times I_l) + \lambda_2 \{(I_m \times K_l) + (K_m \times K_l)\}$$

Also let $B_1 = K_m \times I_l,\ B_2 = (I_m \times K_l) + (K_m \times K_l)$

Then $B_1^2 = (m-1) B_0 + (m-2) B_2$

and $B_2^2 = m(l-1)(I_m \times I_l) + m(l-2)(K_m \times I_l) + m(l-2)(J_m \times K_l)$

$\therefore$ $n_1 = m - 1,\ n_2 = m(l-1)$

$$B_1 B_2 = (m-1) B_1$$

The above products give the values of p_{jk}^i; $(0 \le i, j, k \le 2)$ which are the parameters of a GD association scheme. Hence the row GOCM is the incidence matrix of a GD design.

Remark : A row GOCM N is BIBD if $\lambda_1 = \lambda_2 = \lambda_3$ and if the column sum of N is k.

CONSTRUCTION OF CERTAIN COMBINATORIAL DESIGN FROM GOCM

3.1. CONSTRUCTION OF BIBD FROM GOCM

Theorem 3. For any prime p there exist a 2-Design with parameters

$$(p^{2n}, p^{2n}(p^n + 1), P^{2n} - 1, p^n - 1, p^n - 2)$$

Proof. For $n > 1$ we give a method of construction using finite field.

Consider the finite the field $GF(p^n)$, $= GF(q)$, $q = p^n$

Let $\{0, 1, x, x^2, \ldots, x^{q-1}\}$ be the elements of $GF(q)$ we construct $(q-1)$ matrices of order q

$(A_1, A_2, \ldots A_{q-1})$ as follows, writing $p^n = q$

Each row and column of $A_s = (a_{ij})$ is headed by $[0, x^S, x^{S+1}, \ldots, x^{q-1}, x, x^2, \ldots, x^{S-1}]$ respectively, the rest of entries are given by $a_{ij} = a_{i1} - a_{1j},\ i, j > 1$.

$\because$ $\{0, x, x^2, \ldots, x^{q-1}\}$ from a finite group each row as well as column are distinct permutations of $[0, x, x^2 \ldots x^{q-1}]$ *i.e.* in any row an element does not appears more than ones. Now replace 0 by null matrix O of order p^n and x^s by α^S where α^S is a (0, 1) circulant matrix such that $\alpha^q = \mathrm{I}$, $\alpha = \mathrm{circ}\ (010 \ldots 0)$.

Next we construct the block matrix.

$$G = [\widetilde{K}\ A_0, A_1, A_2, \ldots A_q]_{q2 \times q2(q+1)} \text{ with}$$

$$\widetilde{K} = \mathrm{circ}\ (K\ 0 \ldots\ldots 0)$$

$$A_0 = \mathrm{circ}\ (0\ \ I \ldots I)$$

It can be verified that G is a row GOCM and is the incident matrix of the B1BD

$$(p^{2n}, p^{2n}\ (p^n + 1), P^{2n} - 1, p^n - 1, p^n - 2)$$

Theorem 4. If $[d_1, d_2, \ldots d_k]$ mod (mn) is a difference set, then

$$N = [\alpha^{d1} + \alpha^{d2} + \ldots.\ \alpha^{dk}] \text{ is a GOCM where } \alpha^{mn} = I,\ \alpha = \mathrm{Circ}\ (0\ 1\ .\ .\ .\ 0\ )$$

and N is the incidence matrix of a BIBD.

Proof. It can be easily verified.

For particular cases see appendix I, table 1.1

H.N.-1, 3, 5, 7, 12, 16, 25, 30, 31, 34, 37, 50, 69, 75, 103, 150, 153, 166, 207

(Combinatorial design theory, Hall [7]).

Theorem 5. If $[d_{11}, d_{12}, \ldots., d_{1k}]$, $[d_{21}, d_{22}, \ldots.., d_{2k}]$, ... $[d_{i1}, d_{i2}, \ldots., d_{ik}]$ mod (mn) is a supplementary difference set, then $N = \left[\sum_{j=1}^{k} \alpha^{d1j} \quad \sum_{j=1}^{k} \alpha^{d2j} \ \ldots\ \sum_{j=1}^{k} \alpha^{dij}\ \right]$ is a GOCM where $\alpha^{mn} = I$, $\alpha = \mathrm{Circ}\ (0\ 1 \ldots 0\ )$ which is the incidence matrix of a BIBD.

Particular cases are

H.N. 9, 20, 29, 42, 56, 57, 58, 60, 85, 86, 92, 93, 95, 101, 103, 108, 124, 125, 154, 155, 157, 185, 186, 188, 187, 190, 197.

(Combinatorial design theory, Hall [7]).

3.2. CONSTRUCTION OF NON SYMMETRIC BIBD

Definition 1(a) : Two rectangular matrices of same size $(m \times n)$ A and B will be called disjoint if their Hadamard product is zero *i.e.*, $A.B = 0$.

1(b) : n rectangular $m \times n$ matrices $A_1, A_2, \ldots A_n$ will be called disjoint if $A_i.A_j = 0_{m \times n}$ for $i, j = 1, 2, \ldots, n;\ i \neq j$.

Definition 2. Rectangular Algebra of (0, 1) matrices. Let $A_1, A_2, \ldots, A_r$ be (0,1) matrices which are disjoint $m \times n$ matrices. Let Ω be a generalized association scheme of square (0, 1) matrices. $A_1, A_2, \ldots, A_r$ is said to constitute a rectangular algebra based on Ω if A_iA_j is a linear combination of association matrices of Ω.

Remark: Rectangular algebra is non associative.

Definition 3. Partial Association scheme

Let R_i be relation from A to B, $i = 1, 2, \ldots, r$

i.e., $$R_i \subseteq A \times B,\ i = 1, 2, \ldots, r$$

If A has m element and B has n elements, then adjancy matrix of R_i is an $m \times n$ matrix B_i defined as $B_i = [\alpha_{jk}]$, where $\alpha_{ij} = \begin{cases} 1, & \text{if } (j,k) \in Ri \\ 0, & \text{otherwise} \end{cases}$

$B_1, B_2, \ldots, B_r$ will be said to constitute a partial association scheme over a generalized association scheme Ω if $B_1, B_2, \ldots, B_r$ constitutes a rectangular algebra over Ω.

i.e., if (i) $B_1, B_2, \ldots, B_r$ are disjoint

(ii) $B_i B_j^T$ is a linear combination of association matrices of Ω.

Remark : (i) Since $B_1 + B_2 +, \ldots, + B_r \neq J_{m,n}$ the association scheme is called partial association scheme

Algorithm for construction of row GOCM from partial association scheme

Step 1. Construct a partial association scheme $B_1, B_2, \ldots, B_r$ of $m \times n$ matrices from a given generalized association scheme Ω of $m \times m$ square matrices $A_1, A_2, \ldots A_q$.

Step 2. Represent the partial association scheme $B_1, B_2, \ldots, B_r$ as

$$a_1B_1 + a_2B_2 + \ldots\ldots + a_rB_r$$

Step 3. Replace $a_1, a_2, \ldots, a_r$ by suitable $p \times p$ (0, 1) matrices $P_1, P_2, \ldots P_r$ such that $P_1 + P_2 +, \ldots, + P_r = J_p$ and adjoin new column of I_p, K_p to construct the $mp \times (n + s)\, p$ matrix N, where s is the number of new columns.

Step 4. N is a row GOCM.

Example : Let Ω be a generalized association scheme (circulant *AS*) defined by the 3 × 3 matrices

$$I_3,\ \omega = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},\quad \omega^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and let

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

we have

$$B_1B_1^T = B_2B_2^T = I,\ B_3B_3^T = 2I$$

$$B_1B_2^T = O,\ B_1B_3^T = \omega = B_2B_3^T$$

Hence (B_1, B_2, B_3) defines a partial association scheme over Ω.

The partial association scheme can be represented as

$$a_1B_1 + a_2B_2 + a_3B_3 = \begin{bmatrix} a_1 & a_2 & 0 & 0 & a_3 & a_3 \\ a_3 & a_3 & a_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & a_3 & a_1 & a_2 \end{bmatrix}$$

If we replace a_1 by $\alpha+\alpha^4$, a_2 by $\alpha^2+\alpha^3$ and a_3 by I_5 where $\alpha^5 = I_5$ and adjoining a new column $\begin{bmatrix} I_5 \\ I_5 \\ I_5 \end{bmatrix}$,

We have the GOCM

$$N = \begin{pmatrix} \alpha+\alpha 4 & \alpha 2+\alpha 3 & O & O & I & I & I \\ I & I & \alpha+\alpha 4 & \alpha 2+\alpha 3 & O & O & I \\ O & O & I & I & \alpha+\alpha 4 & \alpha 2+\alpha 3 & I \end{pmatrix}$$

Clearly N is row column regular and $R_iR_J = I + K$, $i \neq j$ and $R_i^2 = 4I + K$

Here $\lambda_1 = \lambda_2 = \lambda_3 = I$

Hence N is type I GOCM giving the incidence matrix of a non symmetric BIBD with parameters $v = 15$, $b = 35$, $r = 7$, $k = 3$, $\lambda = 1$.

3.3. CONSTRUCTION OF RD & GD FROM GOCM

Theorem 7. If a (0, 1) regular square matrix A satisfying

$$A^2 = P_{11}^0 I + P_{11}^1 A + P_{11}^2 (J - I - A)$$

Then A is a $(v, P_{11}^0, P_{11}^1, P_{11}^2)$ strongly regular graph and gives a GD design.

Proof. Let I, K be the two usual (0, 1) matrices.

And let $A = K \times I + I \times K$

$$= \begin{pmatrix} K & I & I & \dots & I \\ I & K & I & & I \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ I & \cdots & \cdots & & K \end{pmatrix}$$

then A is a 2-class *AS* or *SRG* satisfying $A^2 = P_{11}^0 I + P_{11}^1 A + P_{11}^2 (J - I - A)$

If K and I is of order n then $K^2 = (n-1)I + (n-2)K$

$$R_j^2 = 2(n-1)I + (n-2)K$$

and

$$R_{ij} = (n-2)I + 2K$$

Therefore A is a row GOCM with parameter $l = s = n$, $m = n = n$, $r = 2(n-1)$,

$$\lambda_1 = n-2\ ,\ \lambda_2 = n-2\ ,\ \lambda_3 = 2$$

and the related design is a RD design wih parameter $v = n^2 = b$, $n_1 = n-1$, $n_2 = n-1$, $n_3 = (n-1)^2$, $\lambda_1 = n-2$, $\lambda_2 = n-2$, $\lambda_3 = 2$

Theorem 8. Let N_{ij} $i, j = 1, 2, \ldots\, n$ be $n \times m$ matrices with entries (0, 1).

Let $N = (N_{ij})$ $i = 1, 2, \ldots\, m$ and $j = 1, 2, \ldots\, s$ be an $m \times s$ block matrix which is a GOCM such that

$$R_i^2 = rI + \lambda_1 K \qquad \ldots (1)$$

$$R_iR_j = \lambda_2 I + \lambda_3 K \qquad \ldots (2)$$

Then N is the incidence matrix of a RD based on the rectangular association scheme represented by the array

1	2	3 - - - - -	n
$n+1$	$n+2$	- - - - - -	$2n$
$2n+1$	$2n+2$	- - - - - -	$3n$
- - -	- - - -	- -	
- - -	- - - - -	-	
- - -	- - - - - -	-	
$(m-1)n+1$	$(m-1)n+2$	- - -	mn.

Proof. Let a pair of points belonging to same row is 1st associate, a pair of points belonging to same column is 2nd associate and other pairs are 3rd associates. However replications of the points and different block sizes of the RD may be different. Also

If $\sum_{j=1}^{s} N_{ij}$ is row regular, for each i with row sum r ----------- (A) and $\sum_{i=1}^{s} N_{ij}$ is column regular for each j with column sum k------(B) then M is the incidence matrix of a RD with parameters $v = mn$, $b = ns$, $r = r$, $k = k$, $\lambda_1, \lambda_2, \lambda_3$, $n_1 = n - 1$, $n_2 = m - 1$, $n_3 = (n - 1)(m - 1)$. The above RD is a GD if $\lambda_1, = \lambda_3$ or $\lambda_2 = \lambda_3$ and $n_1 = (n_1 + n_3)$ if $\lambda_{1,} = \lambda_3$, or $n_2 = (n_2 + n_3)$ if $\lambda_2, = \lambda_3$.

Also the RD is a BIBD if $\lambda_1 = \lambda_2 = \lambda_3$.

Remark: The following parametric relations must be satisfied

(i) $nr = ks$ and

(ii) $\lambda_1 (n - 1) + \lambda_2 (m - 1) + \lambda_3 (n - 1)(m - 1) = r(k - 1)$.

(*) If (A) and (B) are dropped and $\lambda_1 = \lambda_2 = \lambda_3$ then the above design is a pair wise balanced design, PBD.

Example. The GOCM N given by

$$N = \begin{pmatrix} I_n & K \\ K & I_n \end{pmatrix} = \begin{pmatrix} 1 & \alpha + \alpha^2 \ ... + \alpha^{n-1} \\ \alpha + \alpha^2 \ .... + \alpha^{n-1} & I \end{pmatrix}$$

where $K = J - I_n$

is a RD based on the scheme

$$\begin{matrix} 1 & 2 & 3 & - & - & - & n \\ n+1 & n+2 & n+3 & - & - & - & 2n \end{matrix}$$

Here $m = 2$

and $R_i^2 = nI + (n - 2)K$

$R_i R_j = 0I + 2K, \quad i, j = 1, 2$

N is the incidence matrix of an RD with parameters

$$v = 2n = b,\ r = k = n,\ \lambda_1 = (n - 2),\ \lambda_2 = 0,\ \lambda_3 = 2,\ n_1 = (n - 1),\ n_2 = 1,\ n_3 = (n - 1).$$

Theorem 9. Let I_{2v}, A_1, A_2 be the association matrix of the partial geometry obtained from the dual of the BIBD ($v = 2k^2\ 2k + 1, k, 2$) then $I_{2v} + \alpha A_1 + \alpha^2 A_2$ is a GOCM. If we replace α and α^2 by circulant matrix we get an RD with $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$.

Proof. IT can be easily verified.

Example

$$M = \begin{pmatrix} 1 & \alpha & \alpha^2 & 1 & \alpha^2 & \alpha \\ \alpha^2 & 1 & \alpha & \alpha & 1 & \alpha^2 \\ \alpha & \alpha^2 & 1 & \alpha^2 & \alpha & 1 \end{pmatrix}$$ where, $\alpha^3 = I$ is a RD design with parameters

$v = 9$, $b = 18$, $r = 6$, $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 3$, $n_1 = 2$, $n_2 = 2$, $n_3 = 4$.

as $R_i^2 = 6I + 0K$, $i = 1, 2, 3$ and $R_{ij} = 0I + 3K$, $i, j = 1, 2, 3$ and $i \neq j$.

Example of RD designs

(1) Let $\alpha = \text{cir}(0\ 1\ 0\ 0\ 0)$ such that $\alpha^5 = I$ and $w = \text{cir}(0\ 1\ 0)$, $w^3 = I$

Then $N = [w + w^2] \times [\alpha + \alpha^2 + \alpha^3 + \alpha^4]$ is a GOCM.

We have $NN^T = [w + w^2]^2 \times [\alpha + \alpha^2 + \alpha^3 + \alpha^4]^2$

$= [2I + J] \times [\alpha + \alpha^2 + \alpha^3 + \alpha^4 + 2\,[\alpha^3 + \alpha^4 + I + I + \alpha + \alpha^2]]$

$= [2I_3 + K_3] \times [4I_5 + 3K_5]$

$= 8I_{15} + 4K_3 \times I_5 + 6I_3 \times K_5 + 3K_3 \times K_5$

which is the incidence matrix of an RD with parameters $v = b = 15$, $r = k = 8$, $\lambda_1 = 4$, $\lambda_2 = 6$, $\lambda_3 = 3$, $n_1 = 2$, $n_2 = 4$, $n_3 = 8$.

Theorem 9. If A, B, C are commutative (0, 1) matrices, then

$$M = \text{Circ}\,(A\ \ B\ \ B\ \ C\ \ B\ \ C\ \ C) = \begin{bmatrix} A & B & B & C & B & C & C \\ C & A & B & B & C & B & C \\ C & C & A & B & B & C & B \\ B & C & C & A & B & B & C \\ C & B & C & C & A & B & B \\ B & C & B & C & C & A & B \\ B & B & C & B & C & C & A \end{bmatrix}$$

is a GOCM if

(i) $B^2 + C^2 + 3BC + A\,(B + C) = (B + C)^2 + AB + BC + CA = \lambda_2 I + \lambda_3 K$

(ii) $A^2 + 3B^2 + 3C^2 = rI + \lambda_1 K$

The condition is satisfied if $A = J$, B, C are association matrices satisfying $P^1_{12} = P^2_{12}$

Then $$R_i \cdot R_i = R_i^2 = (3v - 2)I_v + (v - 2 - 2P^1_{12})K$$

and $$R_i \cdot R_j = (v - 1)I_v + (v + P^1_{12} - 1)K$$

Remark: (1) This is an RD with parameters

$$v' = 7v = b, r = k = 3v - 2$$

$\lambda_1 = v - 2 - 2P^1_{12}$, $\lambda_2 = v - 1$, $\lambda_3 = v - 1 + P^1_{12}$, $n_1 = 6$, $n_2 = v - 1$, $n_3 = 6\,(v - 1)$.

Remark: (2) When $A = K = \alpha + \alpha^2$, and $B = C = \alpha$ and α = cir (010) then GOCM is a GD design with parameters $v = 21 = b$, $r = k = 8$, $\lambda_1 = 1$, $\lambda_1 = 7$, $n_1 = 6$, $n_2 = 14$

3.4. CONSTRUCTION OF (r, λ)-DESIGN FROM ORTHOGONAL ARRAY OF STRENGTH TWO AND ROW GCOM.

Theorem 10. Let I and K are of order 4 and l, s, p, q, t are positive integers, such that $t = 2q - p$, then there is a (r, λ)-design with parameter $v = 4l$, $b = 4s$, $r = 3p + 4q + t$, $\lambda = 2\,(p + q)$.

Proof. Suppose I and K are of order n and p, q, t are positive integers. We construct an array N with l rows and s columns of I and K with each row having $(p + q)$ Ks and $(t + q)$ is such that in the array typical arrangements of any two rows are

$K...K$ (q times) $I...I$ (q times) $I...I$ (t times) $K...K$ (p times)

$I...I$ (q times) $K...K$ (q times) $I...I$ (t times) $K...K$ (p times)

then $$R_i^2 = (p + q)\,[(n - 1)\,I + (n - 2)\,K] + (t + q)\,I \quad \dots (1)$$

and $$R_{ij} = (t + pn - p)\,I + [p\,(n - 2) + 2q]\,K \quad \dots (2)$$

Clearly the array is a row GOCM

For the row GOCM, to be an $(r, \boldsymbol{\lambda})$-design

We must have $$t = 2q - p \quad \dots (3)$$

$$n = 4 \quad \ldots (4)$$

$$\lambda = 2\,(p + q)$$

i.e. I and K are 4×4 matrices. Hence the theorem.

Display 1 : For $l = 4$, $s = 4$, $q = 1$, $p = 2$, $t = 0$ each row contains $p + q = 3$ K's and $t + q = 1$ I's.

Consider the block matrix $N = \begin{bmatrix} I & K & K & K \\ K & I & K & K \\ K & K & I & K \\ K & K & K & I \end{bmatrix}$

We have
$$R_i^2 = I + 3K^2 = I + 3\,[3I + 2K]$$
$$= 10I + 6K$$
$$R_{ij} = 2K^2 + 2K = 2\,[3I + 2K] + 2k$$
$$= 6I + 6K$$

The (r, λ)-design is a BIBD with parameters (16, 16, 10, 10, 6).

3.6. (r, λ)-DESIGN FROM BIBD

Theorem11. If there is a BIBD (v, b, r, k, λ) then by substituting I for 0 and K for 1 in the incidence matrix of BIBD and adding t numbers of *Is* and s number *Ks* there exists an (r, λ)-design with parameters $v' = 4v$, $b' = 16\,(r - \lambda)$, $r' = 4\,(r - \lambda) + 2(r + s)$, $\lambda' = 2\,(r + s)$.

Proof. If I and K are of size n then we have
$$R_i^2 = rK^2 + (b - r)\,I$$
$$= [r\,(n - 1) + (b - r)]\,I + (n - 2)\,r\,K$$
$$R_{ij} = \lambda K^2 + (b - 2r + \lambda)\,I + (b - [b - 2r + 2\lambda])\,K$$
$$= [b - 2r + n\,\lambda]\,I + [\lambda\,(n - 4) + 2r]\,K$$

For (r, λ)-design
$$(n - 2)\,r = b - 2r + n\,\lambda = \lambda\,(n - 4) + 2r \quad \ldots (1)$$
$$nr = b + 2n, \;\; n = \frac{b}{r + \lambda} \quad \ldots (2)$$
$$(n - 4)\,r = \lambda\,(n - 4), \qquad r = \lambda \text{ is trivial when } n = 4$$

Now we add t numbers of I's and s number K then
$$R_{ij} = [b - 2r + n\,\lambda]\,I + [\lambda\,(n - 4) + 2r]\,K + tI + s\,[(n - 1)\,I + (n - 2)\,K]$$
$$= [b - 2r + n\,\lambda + t + s\,(n - 1)]\,I + [\lambda\,(n - 4) + 2r + s\,(n - 2)]\,K$$
$$R_i^2 = [r\,(n - 1) + (b - \mathrm{r})]\,I + (n - 2)\,rK + tI + s\,[(n - 1)\,I + (n - 2)\,K]$$
$$R_i^2 = [r\,(n - 1) + (b - r) + t]\,I + (n - 2)\,(r + s)\,K$$

For (r, λ)-design
$$(n - 2)\,(r + s) = b - 2r + t + n\,(\lambda + s) - s$$
$$= \lambda\,(n - 4) + 2r + s\,(n - 2) \quad \ldots (1)$$
$$(n - 4)\,r = \lambda\,(n - 4) \quad \Rightarrow n = 4 \quad \ldots (2)$$

And
$$b - 2r + t + n\,\lambda + ns - s = n\,\lambda - 4\,\lambda + 2r + sn - 2s$$

$$b - 4r + 4\lambda + s + t = 0$$

$$b + s + t = 4\,(r - \lambda)$$

$$s + t = 4\,(r - \lambda) - b \geq 0 \qquad \ldots(3)$$

Note : When $4\,(r - \lambda) = b$, then design is BIBD.

Remarks : In all above design $s + t$ is small

i.e., $s + t = 0$ or 1or 2.

Result : When $s + t = 4\,(r - \lambda) - b \geq 0$ and $n = 4$.

The (r, λ)-design obtained from BIBD (v, b, r, k, λ) by the construction theorem has parameters $v' = 4v$, $b' = 16(r - \lambda)$

$$r' = b + 2r + 3s + t = 4\,(r - \lambda) + 2\,(r + s),\ \lambda' = 2(r + s)$$

With
$$\text{Max. } v' = \frac{r'' - \lambda'}{\frac{(r')^2}{b'} - \lambda'} = \frac{b + 2r + 3s + t - 2\,(r+s)}{\frac{(b + 2r + 3s + t)^2}{16\,(r - \lambda)}} - 2\,(r + s)$$

$$\text{Max. } V' = \frac{16\,(r - \lambda)^2}{(r - s - 2\lambda)^2}$$

Remark : Our designs are non-near trivial and irreducible.

Remark : We classify the (r, λ)-design.

If $\lambda\,(v - 1) - r\,(r - 1) < 0$, elliptical

$= 0$, parabolic

> 0, hyperbolic.

Some examples of (r, λ)-design from BIBD

(1) BIBD (7, 3, 1) H.N. 1

$v = 7$, $b = 7$, $r = 3$, $\lambda = 1$, $(r - \lambda) = 2$ and $s + t = 4\,(r - \lambda) - b = 1$

Case (i). $s = 0$, $t = 1$, the (r, λ)-design is

$v' = 4v = 28$, $b' = 16\,(r - \lambda) = 32$, $r' = 4\,(r - \lambda) + 2\,(r + s) = 14$, $\lambda' = 2\,(r + s) = 6$

Case (ii). $s = 1$, $t = 0$, the (r, λ)-design is

$v' = 4v = 28$, $b' = 16\,(r - \lambda) = 32$, $r' = 4\,(r - \lambda) + 2(r + s) = 16$, $\lambda' = 2\,(r + s) = 8$

Remark : The (r, λ)-design is a *DK* design ($r^2 = \lambda b$).

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