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A NOTE ON TORSIONAL VIBRATIONS IN A NON-HOMOGENEOUS COMPOSITE SHAPE VISCO-ELASTIC CIRCULAR CYLINDER

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In this paper, we discuss the problem of free torsional vibrations of a non-homogeneous composite shape visco-elastic solid circular cylinder, whose inner core is of Achenbech and Chao type visco-elastic material, whereas outer surface is of visco-elastic material of general linear type. Bessel functions have been used as a tool to solve the problem. Such type of problems are of importance in engineering structural mechanics, materials science, Geophysics etc.

Introduction

Ghosh [1] discussed vibrations of non-homogeneous spherically aeolotropic spherical shell. Johnson [2] dealt with radial vibrations of an isotropic spherical shell. Abdou & Khamis [3] discussed the problem of an infinite plate with a curvilinear hole having three poles & arbitrary shape. Lal [4] gave a note on the torsional body forces in visco-elastic half space. Dey & Charavorty [5] found the influence of gravity and initial stress on Love waves in a transversely isotropic medium. Sengupta & Mahapatra [6] investigated torsional vibrations of a composite cylinder. Mahapatra [7] discussed torsional vibratons of an orthotropic cylindrical shell having periodic shearing forces Dey and Mahto [8] discussed Surface waves in a highly pre-stressed medium. Sharma & Chand [9] attempted to discuss the vibrations in a transversely isotropic plate due to sudden punching of hole. Kang [10] studied vibrations of complete hollow spheres with variable thickness. Ponnusamy & Selvamani [11] investigated wave propagation in a homogeneous isotropic cylindrical panel embedded on elastic media. In this paper, we discuss the problem of free torsinal vibrations of a non-homogeneous composite shape visco-elastic solid circular cylinder, whose inner core ($0 \leq r \leq b$) is of Achenbech and Chao type visco-elastic material, whereas outer surface ($b \leq r \leq a$) is of visco-elastic material of general linear type. Such type of problems are of importance in engineering structural mechanics, materials science, Geophysics etc.

Let the variations in elastic constants and densities are taken as,

$$G_1 = G_0(1 + kz)^n \quad \text{... (1.1)}$$

$$E_2 = E'_2(1 + kz)^n \quad \text{... (1.2)}$$

$$\rho_j = \rho'_j(1 + kz)^n \quad \text{... (1.3)}$$

($\boldsymbol{j}$ = 1, 2)

where G_0, E_2', ρ'_j, k are the constants and n is an integer. Bessel functions have been used as a tool to solve the problem.

Formulation of the problem, boundary & continuity conditions

Let us consider a composite visco-elastic solid circular cylinder whose inner core ($0 \leq r \leq b$) is of Achenbech and Chao type viscoelastic material, whereas outer surface ($b \leq r \leq a$) is of viscoelastic material of general linear type.

Making use of cylindrical polar coordinates (r, θ, z) and assuming that axis of the cylinder coincides with z-axis, the components of displacement become (cf [12]),

$$u_j = w_j = 0 \qquad \dots (2.1)$$

$$v_j = V_j(r,z)\, e^{ipt}.$$

With the help of (1.1), (1.2) and (1.3), the non-vanishing stress components for visco-elastic material of general linear type are (cf [13]),

$$\sigma_{r\theta_1} = 2G_0Q_1(1+kz)^n\left(\frac{\partial V_1}{\partial r} - \frac{V_1}{r}\right)e^{ipt} \qquad \dots (2.2)$$

$$\sigma_{\theta z_1} = 2G_0Q_1(1+kz)^n\frac{\partial V_1}{\partial z}\, e^{ipt}$$

and non-vanishing stress components of Achenbech and Chao type visco-elastic material are (cf [14]),

$$\sigma_{r\theta_2} = E_2{'}Q_2(1+kz)^n\left(\frac{\partial V_2}{\partial r} - \frac{V_2}{r}\right)e^{ipt} \qquad \dots (2.3)$$

$$\sigma_{\theta z_2} = E_2{'}Q_2(1+kz)^n\frac{\partial V_2}{\partial z}\, e^{ipt}$$

where

$$Q_1 = \left(\frac{1 + i\,\vartheta_1 p}{1 + i\,\vartheta_2 p}\right),$$

$$Q_2 = E'\left(\frac{s + ip}{k' + i\,p}\right)^2.$$

The two stress-equations are identically satisfied. The only non-vanishing stress-equations of motion are (cf [12]),

$$\rho_j\frac{\partial^2 v_j}{\partial t^2} = \frac{\partial(\sigma_{r\theta})_j}{\partial r} + \frac{\partial(\sigma_{\theta z})_j}{\partial z} + 2\,\frac{(\sigma_{r\theta})_j}{r} \qquad \dots (2.4)$$

$$(\boldsymbol{j} = \mathbf{1}, \mathbf{2})$$

The boundary condition is,

$$\sigma_{r\theta_1} = -Pe^{ipt} \text{ on } r = a \qquad \dots (2.5)$$

and continuity condition is,

$$\sigma_{r\theta_1} = \sigma_{r\theta_2} \text{ for } r = b \qquad \dots (2.6)$$

Solution of the problem

Putting (1.1)-(1.3) and (2.1)-(2.3) in (2.4), we get,

$$\left[\frac{\partial^2 V_1}{\partial r^2}+\frac{1}{r}\frac{\partial V_1}{\partial r}-\frac{V_1}{r^2}\right]+\left[\frac{\partial^2 V_1}{\partial z^2}+\left(\frac{nk}{1+kz}\right)\frac{\partial V_1}{\partial z}+\frac{\rho_0 p^2}{2G_0 Q_1}V_1\right]=0 \quad ...(3.1)$$

$$\left[\frac{\partial^2 V_2}{\partial r^2}+\frac{1}{r}\frac{\partial V_2}{\partial r}-\frac{V_2}{r^2}\right]+\left[\frac{\partial^2 V_2}{\partial z^2}+\left(\frac{nk}{1+kz}\right)\frac{\partial V_2}{\partial z}+\frac{\rho'_0 p^2}{Q_2}V_2\right]=0 \quad ...(3.2)$$

Making use of method of separation of variables, we substitute,

$$V_1 = F_1(r)\,H_1(z) \quad \dots(3.3)$$
$$V_2 = F_2(r)\,H_2(z)$$

in (3.1) and (3.2) to get,

$$\frac{1}{F_1}\left[\frac{\partial^2 F_1}{\partial r^2}+\frac{1}{r}\frac{\partial F_1}{\partial r}-\frac{F_1}{r^2}+\frac{p^2}{\alpha_1^2}F_1\right]=-\frac{1}{H_1}\left[\frac{\partial^2 H_1}{\partial z^2}+\left(\frac{nk}{1+kz}\right)\frac{\partial H_1}{\partial z}\right]=-\beta_1^2 \quad ...(3.4)$$

$$\frac{1}{F_2}\left[\frac{\partial^2 F_2}{\partial r^2}+\frac{1}{r}\frac{\partial F_2}{\partial r}-\frac{F_2}{r^2}+\frac{p^2}{\alpha_2^2}F_2\right]=-\frac{1}{H_2}\left[\frac{\partial^2 H_2}{\partial z^2}+\left(\frac{nk}{1+kz}\right)\frac{\partial H_2}{\partial z}\right]=-\beta_2^2 \quad ...(3.5)$$

which split up into four equations, namely

$$\frac{\partial^2 F_1}{\partial r^2}+\frac{1}{r}\frac{\partial F_1}{\partial r}+\left(\mu_1^2-\frac{1}{r^2}\right)F_1=0, \quad ...(3.6)$$

$$\frac{\partial^2 F_2}{\partial r^2}+\frac{1}{r}\frac{\partial F_2}{\partial r}+\left(\mu_2^2-\frac{1}{r^2}\right)F_2=0 \quad ...(3.7)$$

and
$$\frac{\partial^2 H_1}{\partial z^2}+\left(\frac{nk}{1+kz}\right)\frac{\partial H_1}{\partial z}-\beta_1^2 H_1=0, \quad ...(3.8)$$

$$\frac{\partial^2 H_2}{\partial z^2}+\left(\frac{nk}{1+kz}\right)\frac{\partial H_2}{\partial z}-\beta_2^2 H_2=0 \quad ...(3.9)$$

where
$$\mu_1^2=\frac{p^2}{\alpha_1^2}+\beta_1^2,$$

$$\mu_2^2=\frac{p^2}{\alpha_2^2}+\beta_2^2,$$

$\alpha_1^2=[2G_0Q_1/\rho_0]$, $\alpha_2^2=Q_2/\rho'_0$, β_1, β_2 are constants.

The solution of (3.6) and (3.7) is,

$$F_1 = A_1 J_1(\mu_1 r)+B_1 Y_1(\mu_1 r) \quad ...(3.10)$$
$$F_2 = A_2 J_1(\mu_2 r)+B_2 Y_1(\mu_2 r) \quad ...(3.11)$$

where A_1, B_1, A_2, B_2, are constants and J_1 and Y_1 are Bessel functions of order one and kind first and second respectively.

For the finite cylinder, $B_1 = 0$ and $B_2 = 0$.

Thus, we have

$$F_1 = A_1 J_1(\mu_1 r), \quad \dots(3.12)$$
$$F_2 = A_2 J_1(\mu_2 r)$$

We put,

$$x_j=\beta_j\frac{(1+kz)}{k}, \quad ...(3.13)$$

in equations (3.8) and (3.9) to get,

$$\frac{d^2H_1}{dx_1^2} + \frac{n}{x_1}\frac{dH_1}{dx_1} - H_1 = 0, \quad \text{... (3.14)}$$

$$\frac{d^2H_2}{dx_2^2} + \frac{n}{x_2}\frac{dH_2}{dx_2} - H_2 = 0 \quad \text{... (3.15)}$$

Putting,

$$H_j = \left(\frac{x_j}{\beta_j}\right)^{\frac{(1-n)}{2}} f_j(x_j), \quad \text{... (3.16)}$$

in (3.14) and (3.15), we have

$$x_1^2\frac{d^2f_1}{dx^2} + x_1\frac{df_1}{dx_1} - [l_1^2 + x_1^2]\, f_1 = 0, \quad \text{... (3.17)}$$

$$x_2^2\frac{d^2f_2}{dx^2} + x_2\frac{df_2}{dx_2} - [l_2^2 + x_2^2]\, f_2 = 0 \quad \text{... (3.18)}$$

where $$l_1 = \frac{(1-n)}{2}, \qquad l_2 = \frac{(1-n)}{2} \quad \text{... (3.19)}$$

The solution of (3.17) and (3.18) become,

$$f_1 = A_3\, I_{l_1}(x_1) + B_3\, K_{l_1}(x_1) \quad \text{...(3.20)}$$

$$f_2 = A_4\, I_{l_2}(x_2) + B_4\, K_{l_2}(x_2) \quad \text{...(3.21)}$$

where I_{l_j} and K_{l_j} are the modified Bessel function of order l_j and j takes the values 1 and 2.

Since v_j is finite as $z \to \infty$, therefore,

$$A_3 = 0 \;\&\; A_4 = 0 \text{ and } f_1 = B_3\, K_{l_1}(x_1)\,,\; f_2 = B_4\, K_{l_2}(x_2) \quad \text{... (3.22)}$$

The complete solution becomes,

$$v_1 = c_1\left(\frac{x_1}{\beta_1}\right)^{\frac{(1-n)}{2}} J_1(\mu_1 r) K_{l_1}(x_1) \quad \text{... (3.23)}$$

$$v_2 = c_2\left(\frac{x_2}{\beta_2}\right)^{\frac{(1-n)}{2}} J_1(\mu_2 r) K_{l_2}(x_2) \quad \text{... (3.24)}$$

where $c_1 = A_1\, B_3$ and $c_2 = A_2\, B_4$.

Again substituting the value of v_1 and v_2 in (2.2) and (2.3), we get the components of stresses in first & second medium as,

$$\sigma_{r\theta_1} = 2G_0Q_1C_1k^{\frac{n-1}{2}}(1+kz)^{\frac{n+2}{2}}\left[\mu_1\, J_0(\mu_1 r) - \frac{1}{r}\, J_1(\mu_1 r)\right] k_{l_1}(x_1)e^{ipt} \quad \text{... (3.25)}$$

$$\sigma_{r\theta_2} = E_2'Q_2C_2k^{\frac{n-1}{2}}(1+kz)^{\frac{n+2}{2}}\left[\mu_2\, J_0(\mu_2 r) - \frac{1}{r}\, J_1(\mu_2 r)\right] k_{l_1}(x_2)e^{ipt} \quad \text{... (3.26)}$$

$$\sigma_{\theta z_1} = 2G_0Q_1C_1k^{\frac{n-1}{2}}(1+kz)^{\frac{n+2}{2}} J_1(\mu_1 r)\; \frac{\partial k_{l_1(x_1)}}{\partial z} e^{ipt} \quad \text{... (3.27)}$$

$$\sigma_{\theta z_2} = E_2'Q_2C_2k^{\frac{n-1}{2}}(1+kz)^{\frac{n+2}{2}}\, J_1(\mu_2 r)\; \frac{\partial k_{l_2(x_2)}}{\partial z} e^{ipt} \quad \text{... (3.28)}$$

Substituting (3.25)-(3.28) in (2.5) and (2.6), we get

$$C_1\ (2G_0Q_1)k^{\frac{n-1}{2}}(1+kz)^{\frac{n+2}{2}}\left[\mu_1 J_0(\mu_1 a)-\frac{1}{r}J_1(\mu_1 a)\right]k_{l_1}(x_1)\ \ =-P \qquad \text{... (3.29)}$$

$$C_1(2G_0Q_1)\left[\mu_1 J_0(\mu_1 b)-\frac{1}{r}J_1(\mu_1 b)\right]k_{l_1}(x_1)$$

$$+C_2\ (E_2'Q_2)\left[\mu_2 J_0(\mu_2 b)-\frac{1}{r}J_1(\mu_2 b)\right]k_{l_1}(x_2)=0 \qquad \text{... (3.30)}$$

Solving (3.29) & (3.30), we get,

$$C_1=\frac{\Delta_1}{\Delta}, \qquad \text{... (3.31)}$$

$$C_2=\frac{\Delta_2}{\Delta}$$

where, $\Delta_1=\begin{vmatrix} 0 & -P \\ X_{22} & 0 \end{vmatrix}$

$$\Delta_2=\begin{vmatrix} X_{11} & -P \\ X_{21} & 0 \end{vmatrix}$$

$$\Delta=\begin{vmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{vmatrix}$$

and $X_{11}=(2G_0Q_1)k^{\frac{n-1}{2}}(1+kz)^{\frac{n+2}{2}}\left[\mu_1 J_0(\mu_1 a)-\frac{1}{r}J_1(\mu_1 a)\right]k_{l_1}(x_1)$

$$X_{21}=(G_0Q_1)\left[\mu_1 J_0(\mu_1 b)-\frac{1}{r}J_1(\mu_1 b)\right]k_{l_1}(x_1)$$

$$X_{22}=-\left(E_2'Q_2\right)\left[\mu_2 J_0(\mu_2 b)-\frac{1}{r}J_1(\mu_2 b)\right]k_{l_1}(x_2)$$

Hence the components of stresses for visco-elastic materials of general linear type become,

$$\sigma_{r\theta_1}=2G_0Q_1\left(\frac{\Delta_1}{\Delta}\right)k^{\frac{n-1}{2}}(1+kz)^{\frac{n+2}{2}}\left[\mu_1 J_0(\mu_1 r)-\frac{1}{r}J_1(\mu_1 r)\right]k_{l_1}(x_1)e^{ipt} \qquad \text{... (3.32)}$$

$$\sigma_{\theta z_1}=2G_0Q_1\left(\frac{\Delta_1}{\Delta}\right)k^{\frac{n-1}{2}}(1+kz)^{\frac{n+2}{2}}J_1(\mu_1 r)\ \frac{\partial k_{l_1(x_1)}}{\partial z}e^{ipt} \qquad \text{... (3.33)}$$

and the components of stresses for Achenbech and chao type visco-elastic materials are given by,

$$\sigma_{r\theta_2}=E_2'Q_2\left(\frac{\Delta_2}{\Delta}\right)k^{\frac{n-1}{2}}(1+kz)^{\frac{n+2}{2}}\left[\mu_2 J_0(\mu_2 r)-\frac{1}{r}J_1(\mu_2 r)\right]k_{l_1}(x_2)e^{ipt} \qquad \text{... (3.34)}$$

$$\sigma_{\theta z_2}=E_2'Q_2\left(\frac{\Delta_2}{\Delta}\right)k^{\frac{n-1}{2}}(1+kz)^{\frac{n+2}{2}}J_1(\mu_2 r)\ \frac{\partial k_{l_2(x_2)}}{\partial z}e^{ipt} \qquad \text{... (3.35)}$$

CONCLUSION

For a specific set of materials, if we put the numerical values of the physical parameters, we may find the variation of stress components with time. We shall attempt it in our next paper.

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