Acta Ciencia Indica, Vol. XLI M, No. 2, 105 (2015)

*H**-NORMAL SPACES

NIDHI SHARMA, POONAM SHARMA

Department of Mathematics N.R.E.C. College, Khurja (U.P.), India

AND

NEERAJ KUMAR TOMAR

Department of Mathematics, Surajbhan Degree College Shikarpur-202395 (U.P.), India

RECEIVED : 10 February, 2015

The aim of this paper is to introduce a new class of closed sets called *H**-closed sets in topological spaces. This class is properly placed between the class of h-closed set and generalized *h*-closed set obtained several properties of such a set. We introduced a new class of normal spaces called *H**-normal spaces by using *H**-open sets and obtained several properties of such a space. Moreover, we obtain some new characterizations and preservation theorems of *H**-normal spaces.

2010 AMS Subject Classification : 54D15, 54C08.

KEY WORDS AND PHRASES : *H**-closed, *gH**-closed, *rgH**-closed, *H**-open, *gH**-open, *rgH**-open sets and *H**-normal spaces,

INTRODUCTION

The aim of this paper is to introduce a new concept of normal spaces called *H**- normal spaces by using *H**-open sets and obtained several properties of such a space. Moreover, we obtain some new characterizations and preservation theorems of *H**-normal spaces. Recently, P.G. Patil *et al*. [**6**] introduce and study two new class spaces of $w\alpha$-normal and $w\alpha$-regular spaces by utilizing $w\alpha$-closed sets and obtained some characterization of $w\alpha$-normal and $w\alpha$-regular spaces .Throughout this paper, (X, τ), (Y, σ) spaces always mean topological spaces X, Y respectively on which no separation axioms are assumed unless explicitly stated.

PRELIMINARIES

2.1. Definition. A subset A of a topological space X is called.

1. **Regular closed** [7] if $A = \text{cl}(\text{int}(A))$.
2. **Semi-open [3]** if $A \subseteq \text{cl}(\text{int}(A))$.
3. ***w*-closed [5]** if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is a semi-open in X.
4. **α-closed** [4] if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$.
5. **α*-set** [2] if $\text{int}(\text{cl}(\text{int}(A))) = \text{int}(A)$.

6. ***C*-set** [**2**] if $A = U \cap V$, where U is an open and V is an α*-set in X.
7. ***h*-closed [1]** if s-cl $(A) \subseteq U$ whenever $A \subseteq U$, and U is a w-open in X.
8. ***gh*-closed** if h-cl $(A) \subseteq U$ whenever $A \subseteq U$, and U is a h-open in X.
9. ***rgh*-closed** if h-cl $(A) \subset U$ whenever $A \subset U$ and U is regularly h-open in X.
10. **Regular *h*-open** if there is a regular open set U such that $U \subseteq A \subseteq h$-cl (U).
11. ***hCg*-closed** if h-cl $(A) \subseteq U$ whenever $A \subseteq U$, and U is C-set in X.
12. ***H**-closed** if h-cl $(A) \subseteq U$ whenever $A \subseteq U$ and U is hCg-open set in X.
13. **Regular *H**-open** if there is a regular open set U such that $U \subset A \subset H^*$-cl (U).
14. ***gH**-closed** if H^*-cl $(A) \subseteq U$ whenever $A \subseteq U$ and U is H^*-open in X.
15. ***rgH**-closed** if H^*-cl $(A) \subseteq U$ whenever $A \subseteq U$ and U is regularly H^*-open in X.

The complement of regular closed (resp. α-closed, w-closed, gh-closed, rgh-closed, H^*-closed, gH^*-closed, rgH^*-closed, semi-open, regular h-open, regular H^*-open) set is said to be regular open (resp. α-open, w-open, gh-open, rgh-open, H^*-open, gH^*-open, rgH^*-open, semi-closed, regular h-closed, regular H^*-closed) set. The intersection of all H^*-closed subset of X containing A is called the H^*-closure of A and is denoted by H^*-cl (A). The union of all H^*-open sets contained in A is called H^*-interior of A and is denoted by H^*-int (A). The family of H^*-open (resp. H^*-closed) sets of a space X is denoted by $H^*O\,(X)$ (resp. $H^*C\,(X)$).

2.2. Remark . Every α-closed (resp. α-open) set is H^*-closed (resp. H^*-open) set.

Definitions stated above, we have the following diagram :

closed ⇒ α-closed ⇒ h-closed ⇒ H^*-closed ⇒ gh-closed ⇒ rgh-closed

⇓ ⇓

gH^*-closed ⇒ rgH^*-closed.

However the converses of the above are not true may be seen by the following examples.

2.3. Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then $A = \{c\}$ is h-closed set as well as H^*-closed set but not closed set in X.

2.4. Example. Let $X = \{a, b, c, d, e\}$ and $\tau = \{\phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}, X\}$. Then the set $A = \{a, d, e\}$ is rgh-closed set as well as rgH^*-closed set but not gh-closed set and not gH^*-closed set in X.

2.5. Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then the set $A = \{c\}$ is gH^*-closed set but not closed set in X.

H*-NORMAL SPACES

3.1. Definition. A topological space X is said to be normal (resp. H^*-normal) if for every pair of disjoint closed sets A and B, there exist open (resp. H^*-open) sets U and V such that $A \subset U$ and $B \subset V$.

3.2. Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $A = \{a\}$ and $B = \phi$ are disjoint closed sets, there exist disjoint open sets $U = \{a, c, d\}$ and $V = \{b\}$ such that $A \subset U$ and $B \subset V$. Hence X is normal as well as H^*-normal because every open set is H^*-open set.

3.3. Remark. By the definitions and examples stated above, we have the following diagram :

$$\text{normal} \Rightarrow \alpha\text{-normal} \Rightarrow H^*\text{-normal}.$$

3.4. Lemma. A subset A of a topological space X is rgH^*-open iff $F \subset H^*\text{-int}\ (A)$ whenever F is regularly closed and $F \subset A$.

3.5. Definition 1. A function $f: X \to Y$ is said to be

1. **Almost *rgH**-closed** if for every regular closed set F of X $f(F)$ is πgH^*-closed in Y.
2. ***H**-closed** if for each closed set in X, $f(F)$ is H^*-closed set in Y.
3. **Almost *H**-closed** if $f(A)$ is H^*-closed in Y for each $A \in RC(X)$.
4. **Almost *gH**-closed** if $f(A)$ is gH^*-closed in Y for each $A \in RC(X)$.

3.6. Theorem. A surjection $f : X \to Y$ is a almost rgH^*-closed if and only if for each subsets S of Y and each $U \in RO(X)$ containing $f^{-1}(S)$, there exists a rgH^*-open set V of Y such that $S \subset V$ and $f^{-1}(\text{V}) \subset \text{U}$.

Proof. Necessity. Suppose that f is almost rgH^*-closed. Let S be a subset of Y and $U \in RO(X)$ containing $f^{-1}(S)$. If $V = Y - f(X - U)$, then V is a rgH^*-open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any regular closed set of X. Then $f^{-1}(Y - f(F)) \subset X - F$ and $X - F \in RO(X)$. There exists a rgH^*-open set V of Y such that $Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Therefore, we have $f(F) \supset Y - V$ and $F \subset X - f^{-1}(V) \subset f^{-1}(Y - V)$. Hence we obtain $f(F) = Y - V$ and $f(F)$ is rgH^*-closed in Y. This shows that f is almost rgH^*-closed.

Preservation theorems and other characterizations of *H-normal spaces**

3.7. Theorem. For a topological space X. The following are equivalent:

(a) X is H^*-normal.

(b) For any pair of disjoint closed sets A and B of X, there exists disjoint gH^*-open sets U and V of X such that $A \subset U$ and $B \subset V$.

(c) For each closed set A and each open set B containing A, there exists a gH^* set U such that $\text{cl}\ (A) \subset U \subset H^*\text{-cl}\ (U) \subset B$.

(d) For each closed A and each h-open set B containing A, there exists H^*-open set U such that $A \subset U \subset H^*\text{-cl}\ (U) \subset \text{int}\ (B)$.

(e) For each closed A and each h-open set B containing A, there exists a gH^*-open set G such that $A \subset G \subset H^*\text{-cl}\ (G) \subset \text{int}\ (B)$.

(f) For each h-closed set A and open set B containing A, there exists H^*-open set U such that $\text{cl}\ (A) \subset U \subset H^*\text{-cl}\ (U) \subset B$.

(g) For each h-closed set A and each open set B containing A, there exists a gH^*-open set G such that $\text{cl}\ (A) \subset G \subset H^*\text{-cl}\ (G) \subset B$.

Proof: (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). Since every H^*-open set is gH^*-open it obvious.

(d) $\Rightarrow$ (e) $\Rightarrow$ (c) and (f) $\Rightarrow$ (g) $\Rightarrow$ (c) : Since every closed (resp. open) set is h-closed (resp. h-open), it is obvious.

(c) $\Rightarrow$ (e). Let A be a closed subset of X and B be a h-open set such that $A \subset B$. Since B is h-open and A is closed, $A \subset$ int (A). Then, there exists a gH*-open set U such that $A \subset U \subset H$*-cl $(U) \subset$ int (B).

(e) $\Rightarrow$ (d). Let A be any closed subset of X and B be a h-open set containing A. Then there exists a gH*-open set G such that $A \subset G \subset H$*-cl $(G) \subset$ int (B). Since G is gH*-open, $A \subset H$*-int (G). Put $U = H$*-int (G), then U is H*-open and $A \subset U \subset H$*-cl $(U) \subset$ int (B).

(c) $\Rightarrow$ (g). Let A be any h-closed subset of X and B be an open set such that $A \subset B$. Then cl $(A) \subset B$. Therefore, there exists a gH*-open set U such that cl $(A) \subset U \subset H$*-cl $(U) \subset B$.

(g) $\Rightarrow$ (f). Let A be any h-closed subset of X and B be an open set containing A. Then there exists a gH*-open set G such that cl $(A) \subset G \subset H$*-cl $(A) \subset B$. Since G is gH*-open and cl $(A) \subset G$. We have cl $(A) \subset H$*-int (G), put $U = H$*-int (G), then U is H*-open and cl $(A) \subset U \subset H$*-cl $(U) \subset B$.

3.8. Theorem. If $f : X \rightarrow Y$ is a continuous almost H*****-closed surjection and X is H*****-normal space, then Y is H*****-normal.

Proof. Let A and B be any closed set and B be a open set containing A. Then f^{-1} (A) and f^{-1} (B) are disjoint closed set of X. Since X is H*****-normal, there exists disjoint H*-open set U and V such that f^{-1} $(A) \subset U$ and f^{-1} $(B) \subset V$. Let $G =$ int (cl (int (U))) and $N =$ int (cl (int (V))), then G and N are disjoint regularly open sets of X such that f^{-1} $(A) \subset G$ and f^{-1} $(B) \subset H$. By Theorem 3.6, there exist disjoint rgH*****-open sets K and L of Y such that, $A \subset K$ and $B \subset L$, f^{-1} $(K) \subset G$ and f^{-1} $(L) \subset N$. Since G and N are disjoint, so are K and L. It follows from Theorem 3.6, that Y is H*****-normal.

3.9. Corollary. If $f : X \rightarrow Y$ is a continuous, H*****-closed surjection and X is a H*-normal space, then Y is H*- normal.

Proof. Easy to verify.

3.10. Corollary. If $f : X \rightarrow Y$ is a continuous almost gH*****-closed surjection and X is a H*****-normal space, then Y is H*****-normal.

Proof. Easy to verify.

3.11 Corollary. If $f : X \rightarrow Y$ is a continuous almost rgH*****-closed surjection and X is a H*****-normal space, then Y is H*****-normal.

Proof. Easy to verify.

References

1. Rodrigo, J. Antony Rex, Jansi, A. Hana Selvi and Theodore, Jessie, On h-closed sets in topological spaces, *International Journal of Math. Archive,* **3(3),** 1057-1062 (2012).
2. Hatir, E., Noiri, T. and Yuksel, S., A decomposition of continuity. *Acta Math. Hungar*., **70(1-2)**, 145-150 (1996).
3. Levin, N., Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, **70**, 36-41 (1963).
4. Njastad, O., On some classes of nearly open sets, *Pacific J. Math*., **15**, 961-970 (1965).
5. Pushpalatha, A., Studies on a Generalization of mappings in a topological spaces, *Ph.D. Thesis*, Bharathiar University, Coimbatore (2000).
6. Patil, P. G., Banchali, S. S. and Gonnagar, P.K., $w\alpha$-separation axioms in topological spaces, *J. of New Results in Sciences*, No. **5**, 96-103 (2014).
7. Stone, M .H., Applications of the theory of Boolean rings to general topology, *Trans. Amer. Math. Soc*., **41**, 375-381 (1937).

❑