DEGREE OF APPROXIMATION OF FUNCTION BELONGING TO WEIGHTED $(L_r, \xi(t))$ CLASS BY $(E,q)(\bar{N}, p_n^{\gamma})$-SUMMABILITY MEANS OF IT'S CONJUGATE FOURIER SERIES

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In this research paper, we have proved a theorem on degree of approximation of a function belonging to weighted Lipischitz class by product summability means of it's conjugate Fourier series.

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Introduction

Let Σa_n be a given infinite series with the sequence of partial sums $\{s_n\}$ and let $\{p_n\}$ be a sequence of constants with $p_0 > 0,\ p_n \geq 0$ for $n > 0$ and $P_n = \sum_{v=0}^{n} p_v$. We defines

$$p_n^{\gamma} = \sum_{v=0}^{n} A_{n-v}^{\gamma-1} p_v$$

$$P_n^{\gamma} = \sum_{v=0}^{n} p_v^{\gamma},\ p_{-i}^{\gamma} = p_{-i}^{\gamma} = 0,\ i \geq 1 \qquad \ldots (1.1)$$

where,
$$A_0^{\gamma} = 1,\ A_n^{\gamma} = \frac{(\gamma+1)(\gamma+2)\ldots(\gamma+n)}{n!},\ (\gamma > -1,\ n = 1,2,3\ldots) \qquad \ldots (1.2)$$

The sequence to sequence transformations (BOOS [1])

$$t_n^{\gamma} = \frac{1}{P_n^{\gamma}} \sum_{v=0}^{n} p_v^{\gamma} s_v \qquad (1.3)$$

defines the sequence $\{p_n^{\gamma}\}$ of the $(\bar{N}, p_n^{\gamma})$ mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n^{\gamma}\}$. If

$$t_n^{\gamma} \to s \text{ as } n \to \infty \qquad \ldots (1.4)$$

Then the series Σa_n is said to be $(\bar{N}, p_n^{\gamma})$–summable to s. It is clear that $(\bar{N}, p_n^{\gamma})$-summability method is regular (Boos [1])

For $\gamma = 1$, $(\bar{N}, p_n^{\gamma})$-summability method reduces to $(\bar{N}, p_n)$-summability method. (BOOS [1])

The sequence-to-sequence transformation (Hardy [2])

$$T_n = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} s_v \qquad \ldots (1.5)$$

defines the sequences $\{T_n\}$ of the (E, q) means of the sequences $\{s_n\}$. If

$$T_n \to s \text{ as } n \to \infty \qquad \ldots (1.6)$$

Then the series Σa_n is said to be (E, q)-summable to s. Clearly (E, q)-summability method is regular, (Hardy [2])

Further, the (E, q) transformation of the $(\bar{N}, p_n^{\gamma})$ transformation of $\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} t_k^{\gamma}$$

$$= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k^{\gamma}} \sum_{v=0}^{k} p_v^{\gamma} s_v \right\} \qquad \ldots (1.7)$$

If, $$\tau_n \to s \text{ as } n \to \infty \qquad \ldots (1.8)$$

Then Σa_n is said to be $(E,q)(\bar{N}, p_n^{\gamma})$-summable to s. Further suppose, the summability method $(E,q)(\bar{N}, p_n^{\gamma})$ is assumed to be regular throughout this paper.

For $\gamma = 1$, $(E,q)(\bar{N}, p_n^{\gamma})$-summability method reduces to $(E,q)\,(\bar{N}, p_n)$-summability method and transformation τ_n becomes τ_n^1 define by,

$$\tau_n^1 = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{p_k} \sum_{v=0}^{k} p_v s_v \right\} \qquad \ldots (1.9)$$

Let $f(t)$ be a periodic function with the period 2π and Lebesgue-integrable over $(-\pi, \pi)$.

The Fourier series associated with f at any point x is defined by,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \equiv \sum_{n=0}^{\infty} A_n(x) \qquad \ldots (1.10)$$

And the conjugate series of the Fourier series (1.10) is defined as,

$$\sum_{n=1}^{\infty}\left(b_n \cos nx - a_n \sin nx\right) \equiv \sum_{n=0}^{\infty} B_n\left(x\right) \qquad \ldots (1.11)$$

Let $\overline{s}_n(f:x)$ be the $n-th$ partial sums of (1.11). The L_∞-norm of a function $f:R \to R$ is defined by

$$\|f\|_\infty = \sup\left\{\left|f\left(x\right)\right| : x \in R\right\}$$

and the L_v–norm is defined by,

$$\|f\|_v = \left(\int_0^{2\pi}\left(\left|f\left(x\right)\right|^v\right)\right)^{\frac{1}{v}},\ v \geq 1 \qquad \ldots (1.13)$$

The degree of approximation of a function $f:R \to R$ by a trigonometric polynomial $p_n(x)$ of degree *n* under norm $\|.\|_\infty$ is defined by (Zygmund [8]).

$$\|P_n - f\|_\infty = \sup\left\{\left|p_n\left(x\right) - f\left(x\right)\right| : x \in R\right\} \qquad \ldots (1.14)$$

and the degree of approximation $E_n(f)$ of a function $f \in L_v$ is given by

$$E_n\left(f\right) = \min_{P_n} \|P_n - f\|_v \qquad \ldots (1.15)$$

A function $f \in Lip(\alpha)$, if

$$\left|f(x+t) - f\left(x\right)\right| = O\left(\left|t\right|^\alpha\right),\ 0 < \alpha \leq 1,\ t > 0 \qquad \ldots (1.16)$$

and a function $f \in Lip(\alpha, r)$, if

$$\left(\int_0^{2\pi}\left|f\left(x+t\right) - f\left(x\right)\right|^r dx\right)^{\frac{1}{r}} = O\left(\left|t\right|^\alpha\right),\ 0 < \alpha \leq 1,\ r \geq 1 \qquad \ldots (1.17)$$

Further, a function $f\left(x\right) \in Lip(\xi\left(t\right), r)$, if

$$\left(\int_0^{2\pi}\left|f\left(x+t\right) - f\left(x\right)\right|^r dx\right)^{\frac{1}{r}} = O\left(\xi\left|t\right|\right),\ r \geq 1,\ t > 0 \qquad \ldots (4.1.18)$$

Finally, a function $f \in W\left(L_r, \xi\left(t\right)\right)$, if

$$\left(\int_0^{2\pi}\left|f\left(x+t\right) - f\left(x\right)\right]\sin^{\beta} x\right|^r dx\right)^{\frac{1}{r}} = O\left(\xi\left|t\right|\right),\ \beta \geq 0$$

For the Lipischitz classes, we have that, (Khan [3])

$$Lip\,\alpha \subseteq Lip\left(\alpha, r\right) \subseteq Lip\left(\xi(t), r\right) \subseteq W\left(L_r, \xi\left(t\right)\right) \qquad \ldots (1.19)$$

Here, we use the following Notations through out this paper.

$$\psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\}$$

$$\phi(t)=\frac{1}{2}\{f(x+t)-f(x-t)-2f(x)\}$$

and
$$\bar{k}_n(t)=\frac{1}{\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k^{\gamma}}\sum_{v=0}^{k}p_v^{\gamma}\right\}\times\times\frac{\cos\frac{t}{2}-\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

KNOWN RESULTS

In 2012, Misra *et. al*. [4] have proved the following theorem.

Theorem 2.1 : If f is a 2π-periodic function of the class $Lip\,\alpha$ then degree of approximation by the product $(E,q)(\bar{N},p_n)$ summability means of the conjugate series (1.11) of the Fourier-series (1.10) is given by

$$\left\|\tau'_n-f\right\|_{\infty}=O\left(\frac{1}{(n+1)^{\alpha}}\right),0<\alpha<1$$

where τ'_n is as defined in (1.9).

And, generalizing the Theorem 2.1 in 2012, Paikray *et. al.* [6] have proved the following theorem.

Theorem 2.2 : If f is a 2π periodic function of class $Lip(\alpha,r)$, then degree of approximation by product $(E,\ q)(\bar{N},\ p_n)$ summability mean of the conjugates series (1.11) of the Fourier-series (1.10) is given by

$$\left\|\tau_n^1-f\right\|_{\infty}=O\left(\frac{1}{(n+1)^{\alpha+\frac{1}{r}}}\right),\ 0<\alpha<1,\ r\geq 1$$

where τ_n^1 is defined in (1.9) .

Further, in 2013, generalizing the Theorem 2.2 Misra *et. al*. [5] have proved the following theorem.

Theorem 2.3 : Let $\xi(t)$ be a positive increasing function and f is a 2π-periodic, Lebesgue-integrable function of class $Lip(\xi(t),\ r)\ r\geq 1,\ t>0,$ then the degree of approximation by the product $(E,q)(\bar{N},p_n)$ summability mean of the conjugate series (1.11) of the Fourier-series (1.10) is given by,

$$\left\|\tau'_n-f\right\|_{\infty}=O\left((n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right),\ r\geq 1,$$

where τ_n^1 is defined as in (1.9).

MAIN RESULTS

The purpose of this paper is to generalised the Theorem 2.3 for $W(L_r, \xi(t))$ class by $(E,q)(\bar{N}, p_n^{\gamma})$–summability means in the following form.

Theorem 3.1 : If $f : R \to R$ is a 2π periodic, Lebesgue integrable function over $[-\pi, \pi]$ and belonging to the class $W(L_r, \xi(t)), r \geq 1.$ Then the degree of approximation by the product $(E,q)(\bar{N}, p_n^{\gamma})$ summability means of the conjugate series (1.11) of the Fourier series (1.10) is given by

$$\|\tau_n - f\|_r = O\left((n+1)^{\beta + \frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right) \qquad \ldots (3.1)$$

where τ_n is as defined in (1.7) provided $\xi(t)$ satisfies the following conditions

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t|\phi(t)|}{\xi(t)}\right)^r \sin^{\beta r} t dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \qquad \ldots (3.2)$$

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^r \sin^{\beta r} t dt\right\}^{\frac{1}{r}} = O\left\{(n+1)^{\delta}\right\} \qquad \ldots (3.3)$$

and $\left\{\frac{\xi(t)}{t}\right\}$ be a decreasing sequence. … (3.4)

where, δ is an arbitrary number such that $s(1-\delta)-1 > 0,$ conditions (3.2) and (3.3) hold uniformly in x and where $\frac{1}{r} + \frac{1}{s} = 1,$ such that $1 \leq r \leq \infty.$

LEMMAS

We have need the following lemmas for the proof of our theorem.

Lemma 4.1:

$$|\bar{k}_n(t)| = O(n), 0 \leq t \leq \frac{1}{n+1}$$

Proof : For $0 \leq t \leq \frac{1}{n+1}$, we have $\sin nt \leq n \sin t$, then (Boos [1])

$$\left|\bar{k}_n(t)\right| = \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k^\gamma}\sum_{v=0}^{k}p_v^\gamma\frac{\cos\frac{t}{2}-\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k^\gamma}\sum_{v=0}^{k}p_v^\gamma\frac{\cos\frac{t}{2}-\cos vt.\cos\frac{t}{2}+\sin vt.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k^\gamma}\sum_{v=0}^{k}p_v^\gamma\left(\frac{\cos\frac{t}{2}\left(2\sin^2 v\frac{t}{2}\right)}{\sin\frac{t}{2}}+\sin vt\right)\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k^\gamma}\sum_{v=0}^{k}p_v^\gamma\left(\mathrm{O}\left(2\sin v\frac{t}{2}\sin v\frac{t}{2}\right)+v\sin t\right)\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k^\gamma}\sum_{v=0}^{k}p_v^\gamma\left(\mathrm{O}(v)+0(v)\right)\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{\mathrm{O}(k)}{P_k^\gamma}\sum_{t=0}^{k}p_v^\gamma\right|$$

$$= \mathrm{O}(n)$$

This prove the lemma.

Lemma 4.2:

$$\left|\bar{k}_n(t)\right| = \mathrm{O}\left(\frac{1}{t}\right),\ \text{for}\ \frac{1}{n+1}\leq t\leq\pi$$

Proof : For $\frac{1}{n+1}\leq t\leq\pi$, by Jordon's Lemma, we have $\sin\left(\frac{t}{2}\right)\geq\frac{t}{\pi}$, then

$$\left|\bar{k}_n(t)\right| = \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k^\gamma}\sum_{v=0}^{k}p_v^\gamma\frac{\cos\frac{t}{2}-\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k^\gamma}\sum_{v=0}^{k}p_v^\gamma\frac{\cos\frac{t}{2}-\cos\frac{vt}{2}.\cos\frac{t}{2}+\sin\frac{vt}{2}.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k^{\gamma}}\sum_{v=0}^{k}p_v^{\gamma}\frac{\pi}{2t}\left(\cos\frac{t}{2}\left(2\sin^2 v\frac{t}{2}\right)+\sin v\frac{t}{2}.\sin n\frac{t}{2}\right)\right\}\right|$$

$$\leq \frac{\pi}{2\pi(1+q)^n .t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k^{\gamma}}\sum_{v=0}^{k}p_v^{\gamma}\right\}\right|$$

$$\leq \frac{1}{2(1+q)^n .t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k^{\gamma}}\sum_{v=0}^{k}p_v^{\gamma}\right\}\right|$$

$$= \frac{1}{2(1+q)^n .t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right|$$

$$= O\left(\frac{1}{t}\right)$$

This proves the lemma.

PROOF OF THE THEOREM

Using Reimann-Lebesgue theorem, we have for the n-th partial sum $\bar{s}_n(f:x)$ of the conjugate Fourier-series (1.11) of $f(x)$ as follows (Titchmarsh [7])

$$\bar{s}_n(f:x)-f(x)=\frac{2}{\pi}\int_0^{\pi}\phi(t)\bar{k}n(t)$$

The $(\bar{N}, p_n^{\gamma})$ transform of $\bar{s}(f:x)$ using (1.3) is given by

$$t_n - f(x) = \frac{1}{\pi P_n^{\gamma}}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}p_k^{\gamma}\frac{\cos\frac{t}{2}-\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}dt$$

deonoting the $(E,q)(\bar{N}, p_n^{\gamma})$ transform of $\bar{s}_n(f:x)$ by τ_n, we have

$$\|\tau_n - f\| = \frac{1}{\pi(1+q)^n}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k^{\gamma}}\sum_{v=0}^{k}p_v^{\gamma}\frac{\cos\frac{t}{2}-\sin\left(v+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right\}dt$$

$$= \int_0^{\pi}\phi(t)\bar{k}_n(t)\,dt$$

$$= \left\{\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{0}\right\}\phi(t)\bar{k}_n(t)\,dt$$

$$= I_1 + I_2 \text{ (say)} \qquad \ldots (5.1)$$

Now,

$$I_1 = \int_0^{\frac{1}{n+1}} \phi(t)\,\overline{k}n(t)\,dt$$

Applying Hölder's inequality, we have that

$$|I_1| \le \left[\int_0^{\frac{1}{n+1}} \left\{\frac{t|\phi(t)|\sin^{\beta} t}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)\left|\overline{k}n(t)\right|}{t\sin^{\beta} t}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{n\xi(t)}{t^{1+\beta}}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left(\frac{1}{n+1}\right) O(n)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)}{t^{1+\beta}}\right\}^s dt\right]^{\frac{1}{s}}$$

$$\le O\left(\frac{1}{n+1}\right) O(n+1)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)}{t^{1+\beta}}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right)\left[\int_0^{\frac{1}{n+1}} t^{-(\beta+1)s}\,dt\right]^{\frac{1}{s}}$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right).(n+1)^{\beta+1-\frac{1}{s}}$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right).(n+1)^{\beta+\frac{1}{r}}$$

$$= O\left((n+1)^{\beta+\frac{1}{r}}.\xi\left(\frac{1}{n+1}\right)\right)$$

and, $$I_2 = \int_{\frac{1}{n+1}}^{\pi} \phi(t)\,\overline{k}_n(t)\,dt \qquad \ldots (5.2)$$

Using Hölder's inequality, we have that

$$|I_2| \le \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta}|\phi(t)|\overset{\beta}{\sin t}}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}} \left\{\frac{\bar{k}n(t).\xi(t)^s}{t^{-\delta}\overset{\beta}{\sin t}}\right\}^{\frac{1}{s}} dt\right]^{\frac{1}{s}}$$

$$= \mathrm{O}(n+1)^{\delta}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{-\delta+\beta+1}}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= \mathrm{O}(n+1)^{\delta}\left[\int_{\frac{1}{\pi}}^{n+1}\left\{\frac{\xi\left(\frac{1}{g}\right)}{g^{-(\beta+1)+\delta}}\right\}^s \frac{dg}{g^2}\right]^{\frac{1}{s}}, \text{ taking } t=\frac{1}{g}$$

$$= \mathrm{O}\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\left[g^{(\beta-\delta+1)s-1}\right]^{\frac{1}{s}}\right\}_{\frac{1}{\pi}}^{n+1}$$

$$= \mathrm{O}\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta-\delta+1-\frac{1}{s}}\right\}$$

$$= \mathrm{O}\left\{\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+\frac{1}{r}}(n+1)^{-\delta}\right\}$$

$$= \mathrm{O}\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \qquad \dots (5.3)$$

Hence, combining (5.1) (5.2) and (5.3), we have that

$$\|\tau_n - f\| = \mathrm{O}\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}$$

This completes the proof of the theorem

Corollary

Following results can be derived as a corollary by our theorem.

Corollary 6.1 : If $\beta = 0$ then $W(L_r, \xi(t))$ class reduces to $Lip(\xi(t)), r$ class and the degree of approximation for conjugate Fourier-series by product $(E,q)(\bar{N}_1, p_n^{\gamma})$-summability mean is given by

$$\|\tau_n - f\| = \mathrm{O}\left((n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right) r > 0$$

Corollary 6.2 : For $\beta = 0$ and $\gamma = 1$ the main theorem reduces to theorem 2.3.

Corollary 6.3 : If $\beta = 0$ and $\xi(t) = t^{\alpha}$ then, $W(L_r, \xi(t))$ class reduces to $Lip(\xi(t), r)$ and the degree of approximation for conjugate Fourier-series by product $(E,q)(\overline{N}, p_n^{\gamma})-$ summability mean is given by

$$\|\tau_n - f\| = O\left(\frac{1}{n+1-\frac{1}{r}}\right) r > 0$$

Corollary 6.4 : If $\beta = 0$ and $\xi(t) = t^{\alpha}$ and $\gamma = 1$, then main theorem reduces to theorem 2.2.

Corollary 6.5 : If $\beta = 0$, $\xi(t) = t^{\alpha}$ and $\gamma \to \infty$, then $W(L_r, \xi(t))$ class reduces to Lip_{α} and the degree of approximation for conjugate Fourier-series by product $(E,q)(\overline{N}, p_n^{\gamma})-$ summability mean is given by

$$\|\tau_n - f\| = O\left(\frac{1}{(n+1)}\right), 0 < \gamma < 1$$

Corollary 6.6: If $\beta = 0$ and $\xi(t) = t^{\alpha}$, $\gamma \to \infty$ and $\gamma = 1$ then main theorem reduces to theorem 2.1.

Conclusions

Our theorem have more general results rather than any previous known results, that will be enrich the literature on Approximation theory.

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