Acta Ciencia Indica, Vol. XLI M, No. 2, 81 (2015)

SOFTLY NORMAL TOPOLOGICAL SPACES

M. C. SHARMA

Department of Mathematics, N. R. E. C. College, Khurja-203131, (U. P.), India

AND

HAMANT KUMAR

Department of Mathematics, S. S. (P. G.) College, Shikarpur-203395, U. P. (India)

RECEIVED : 30 January, 2015

REVISED : 19 April, 2015

In the present paper, we introduce a weaker version of normality called soft-normality. We will prove that soft-normality is a property, which is implied by quasi-normality and almost-normality. We prove that soft-normality is a topological property and it is a hereditary property with respect to closed domain subspace.

2010 AMS Subject classification: 54D15

KEYWORDS AND PHRASES: regular closed, π-closed, π-open sets; quasi normal and softly normal spaces.

Introduction

In 1968, the notion of a quasi normality is a weaker form of normality was introduced by Zaitsev [11]. In 1970, the concept of almost normality was introduced by Singal and Arya [6]. In 1973, the notion of mild normality was introduced by Shchepin [8] and Singal and Singal [7] independently. In 1990, Lal and Rahman [4] have further studied notion of quasi normal and mildly normal spaces. In 2008, π-normal topological spaces were introduced by Kalantan [2]. In 2011, Thabit and Kamarulhaili [9] presented some characterizations of weakly (resp. almost) regular spaces. Also object of this paper is to present some conditions to assure that the product of two spaces will be π-normal. In 2012, Thabit and Kamarulhaili [10] introduced a weaker version of p-normality called πp-normality and obtained some basic properties, examples, characterizations and preservation theorems of this property are presented. In 2014, Patil, Benchalli and Gonnagar [5] introduced and studied two new classes of spaces, namely ωα–normal and ωα–regular spaces and obtained their properties by utilizing ωα–closed sets. In 2015, Hamant *et al*. [1] introduce a new class of normal spaces is called $\pi g\beta$–normal spaces, by using $\pi g\beta$-open sets. We proved that $\pi g\beta$-normality is a topological property and it is a hereditary property with respect to π-open, $\pi g\beta$-closed subspace. Further we obtain a characterization and preservation theorems for $\pi g\beta$–normal spaces.

In the present paper, we introduce a weaker version of normality called soft-normality. We will prove that soft-normality is a property, which is implied by quasi-normality and almost-normality. We prove that soft-normality is a topological property and it is a hereditary property with respect to closed domain subspace.

Preliminaries

Throughout this paper, spaces (X, τ), (Y, σ), and (Z, γ) (or simply X, Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by cl (A) and int (A) respectively. A subset A is said to be regular open [3] (resp. regular closed [3]) if $A = \text{int}(\text{cl}(A))$ (resp. $A = \text{cl}(\text{int}(A))$. The finite union of regular open sets is said to be π-open. The complement of a π-open set is said to be π- closed.

Remark. Every regular open (resp. regular closed) set is π–open (resp. π-closed).

Softly-normal spaces

3.1. Definition. A space X is called π–normal [2] if for any two disjoint closed subsets A and B of X, one of which is π-closed, there exist disjoint open sets U and V of X such that $A \subset U$ and $B \subset V$.

3.2. Definition. A space X is called almost normal [6] if for any two disjoint closed subsets A and B of X, one of which is regular closed, there exist disjoint open sets U and V of X such that $A \subset U$ and $B \subset V$.

3.3. Definition. A space X is called quasi normal [11] if for any two disjoint π-closed subsets A and B of X, there exist disjoint open sets U and V of X such that $A \subset U$ and $B \subset V$.

3.4. Definition. A space X is called softly normal if for any two disjoint closed subsets A and B of X, one of which is π-closed and other is regularly closed, there exist disjoint open sets U and V of X such that $A \subset U$ and $B \subset V$.

3.5. Definition. A space X is called mildly normal [7, 8] if for any two disjoint regularly closed subsets A and B of X, there exist disjoint open sets U and V of X such that $A \subset U$ and $B \subset V$.

By the definitions stated above, we have the following diagrams [2]:

normal $\Rightarrow$ π-normal $\Rightarrow$ almost normal $\Rightarrow$ softly normal $\Rightarrow$ mildly normal

and

normal $\Rightarrow$ π-normal $\Rightarrow$ quasi normal $\Rightarrow$ softly normal $\Rightarrow$ mildly normal.

3.6. Theorem. For a space X, the following are equivalent:

(i) X is softly-normal.

(ii) For every π-closed set A and every regularly open set B with $A \subset B$, there exists an open set U such that $A \subset U \subset \text{cl}(U) \subset B$.

(iii) For every regularly closed set A and every π-open set B with $A \subset B$, there exists an open set U such that $A \subset U \subset \text{cl}(U) \subset B$.

(iv) For every pair consisting of disjoint sets A and B, one of which is π-closed and the other is regularly closed, there exist open sets U and V such that $A \subset U$, $B \subset V$ and $\text{cl}(U) \cap \text{cl}(V) = \emptyset$.

Proof.

(i) $\Rightarrow$ (ii). Assume (i). Let A be any π-closed set and B be any regularly open set such that $A \subset B$. Then $A \cap (X - B) = \emptyset$, where $(X - B)$ is regularly closed. Then there exist disjoint open sets U and V such that $A \subset U$ and $(X - B) \subset V$. Since $U \cap V = \emptyset$, then cl $(U) \cap V = \emptyset$. Thus cl $(U) \subset (X - V) \subseteq (X - (X - B)) = B$. Therefore, $A \subset U \subset$ cl $(U) \subset B$.

(ii) ⇒ (iii). Assume (ii). Let A be any regularly closed set and B be any π-open set such that $A \subset B$. Then, $(X - B) \subset (X - A)$, where $(X - B)$ is π-closed and $(X - A)$ is regularly open. Thus by (ii), there exists an open set W such that $(X - B) \subset W \subset$ cl $(W) \subset (X - A)$. Thus $A \subset (X -$ cl $(W)) \subset (X - W) \subset B$. So, let $U = (X -$ cl $(W))$, which is open and since $W \subset$ cl (W), then $(X -$ cl $(W) \subset (X - W)$. Thus $U \subset (X - W)$, hence cl $(U) \subset$ cl $(X - W) = (X - W) \subset B$.

(iii) ⇒ (iv). Assume (iii). Let A be any regular closed set and B be any π-closed set with $A \cap B = \emptyset$. Then $A \subset (X - B)$, where $(X - B)$ is π-open. By (iii), there exists an open set U such that $A \subset U \subset$ cl $(U) \subset (X - B)$. Now, cl (U) is closed. Applying (iii) again we get an open set W such that $A \subset U \subset$ cl $(U) \subset W \subset$ cl $(W) \subset (X - B)$. Let $V = (X -$ cl $(W))$, then V is open set and $B \subset V$. We have $(X -$ cl $(W)) \subset (X - W)$, hence $V \subset (X - W)$, thus cl $(V) \subset$ Cl $(X - W)$ $= (X - W)$. So, we have cl $(U) \subset W$ and cl $(V) \subset (X - W)$. Therefore cl $(U) \cap$ cl $(V) = \emptyset$

(iv) ⇒ (i) is clear.

Using Theorem 3.6, it is easy to show the following theorem, which is a Urysohn's Lemma version for soft normality. A proof can be established by a similar way of the normal case.

3.7. Theorem. A space X is softly-normal if and only if for every pair of disjoint closed sets A and B, one of which is π-closed and other is regularly closed, there exists a continuous function f on X into [0, 1], with its usual topology, such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

It is easy to see that the inverse image of a regularly closed set under an open continuous function is regularly closed and the inverse image of a π-closed set under an open continuous function π-closed. We will use that in the next theorem.

3.8. Theorem. Let X is a softly normal space and $f : X \to Y$ is an open continuous injective function. Then $f(X)$ is a softly normal space.

Proof. Let A be any π-closed subset in $f(X)$ and let B be any regularly closed subset in $f(X)$ such that $A \cap B = \emptyset$. Then $f^{-1}(A)$ is a π-closed set in X, which is disjoint from the regularly closed set $f^{-1}(B)$. Since X is softly normal, there are two disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Since f is 1-1 and open, result follows.

3.9. Corollary. Soft normality is a topological property.

3.10. Lemma. Let M be a closed domain subspace of a space X. If A is an open set in X, then $A \cap M$ is open set in M.

3.11. Theorem. A closed domain subspace of a softly normal space is softly normal.

Proof. Let M be a closed domain subspace of a softly normal space X. Let A and B be any disjoint closed sets in M such that A is regularly closed and B is π-closed. Then, A and B are disjoint closed sets in X such that A is regularly closed and B is π-closed in X. By soft normalily of X, there exist disjoint open sets U and V of X such that $A \subset U$ and $B \subset V$. By the Lemma 3.10, we have $U \cap M$ and $V \cap M$ are disjoint open sets in M such that $A \subset U \cap M$ and $B \subset V \cap M$. Hence, M is softly normal subspace.

Since every closed and open (clopen) subset is a closed domain, then we have the following corollary.

3.12. Corollary. Soft normality is a hereditary with respect to clopen subspaces.

Conclusion

In this paper, we have introduced weak form of normality namely softly-normality and established their relationships with some weak forms of normal spaces in topological spaces.

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