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ON PARA KÄHLER MANIFOLD WITH CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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We consider Para Kähler manifold and study the constant Holomorphic Sectional Curvature. We further study Almost Hermite Manifold and find the Condition that Para Kähler Manifold with Constant Holomorphic sectional Curvature is Semi-Symmetric, Ricci Semi-Symmetric and it is also an Einstein manifold.

KEYWORDS : Almost Hermite Manifold, Holomorphic Sectional Curvature, Semi-Symmetric and Ricci Semi-Symmetric, Para Kähler manifold

INTRODUCTION

Sawaki and Sekigawa [1] and Defever and other [2], and Mishra R.S. [3], Sharma, R. and other [10] have Studied Semi-Symmetric and Ricci-Symmetric Para Kähler manifolds. Henri Anciaux and Nikos Georgoiu [4], Bang Y.C. [5] and Dimitri A., Costantino, M. [6] studied Lagrangian submanifold and Einstein Mattices on homogeneous Manifold in Para Kähler manifolds. Salamon, S. [7], Cobrera, F.M. and other [8], Watanebe, Y. [9] have studied Almost Hermitian Manifold and Para Kähler manifold

Let (M, G, F) be a connected, $n = 2m$-dimensional $(m \geq 2)$ Manifold of class C^{∞} with complex structure tensor F satisfying.

$$\overline{\overline{X}} + X = 0 \qquad \dots (1.1)(a)$$

where

$$\overline{X} \stackrel{\text{def}}{=} FX \qquad \dots (1.1)(b)$$

and a non necessarily definite metric tensor G satisfying :

$$G(\overline{X}, \overline{Y}) = G(X, Y) \qquad \dots (1.2)$$

Then (M, G, F) is called an almost Hermite manifold [7], [8], [9]. The almost Hermite manifold (M, G, F) is said to be a para Käkler manifold [1], [3] if its Riemannian Curvature tensor $'K$ satisfying the Kähler identity.

$$'K(X, Y, Z, U) = {}'K(X, Y, \overline{Z}, \overline{U}) \qquad \dots (1.3)$$

Remark (1.1). In the above and in what follows the letters X, Y, Z, U ……. etc. Denote the C^{∞} vector fields in the Lie algebra $L(M)$ of the manifold M.

We know that every Kähler manifold satisfies (1.3) but converse is not true. In view of the following identities for Riemanniam curvature tensor $'K$.

$$'K(X, Y, Z, U) = -{}'K(Y, X, Z, U) = -{}'K(Z, U, X, Y) = -{}'K(U, Z, X, Y)$$

$$= -\,'K(X, Y, U, Z) \qquad \ldots (1.4)$$

And the equation (1.1), the equation (1.3) gives the following results;

$$'K(X, Y, Z, U) = {}'K(\bar{X}, \bar{Y}, Z, U) = {}'K(X, Y, \bar{Z}, \bar{U}) = {}'K(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}), \qquad \ldots (1.5)(a)$$

and

$$K(X, Y, \bar{Z}, U) + K(X, Y, Z, \bar{U}) = 0 \qquad \ldots (1.5)(b)$$

Para Kähler manifold with constant holomorphic sectional curvature

In an almost Hermite manifolds the holomorphic sectional curvature with regard to any vector field X is given by [3], [7]

$$k(X)\, G(X, X)\, G(X, X) + {}'K(X, \bar{X}, X, \bar{X}) = 0 \qquad \ldots (2.1)$$

Also, we know that the condition that the Almost Hermite manifold be of constant holomorphic sectional curvature is [3]

$$\begin{aligned} & 3'K(X, Y, Z, U) + 3'K(\bar{X}, \bar{Y}, Z, U) + 3'K(X, Y, \bar{Z}, \bar{U}) + {}'K(\bar{X}, \bar{Y}, Z, U) \\ & + {}'K(X, U, \bar{Z}, \bar{Y}) + {}'K(Y, U, \bar{X}, \bar{Z}) + {}'K(Y, Z, \bar{U}, \bar{X}) + {}'K(Z, X, \bar{U}, \bar{Y}) \\ & + {}'K(X, Z, \bar{U}, \bar{Y}) + {}'K(Y, \bar{U}, Z, \bar{X}) + {}'K(\bar{U}, X, Z, \bar{Y}) + {}'K(Z, \bar{Y}, U, X) \\ = & -4K\{2'F(X, Y)'F(Z, U) + G(X, Z)\, G(Y, U) - G(Y, Z)\, G(X, U) + {}'F(Y, Z)\, 'F(U, X) \\ & + {}'F(X, Z)'\, F(Y, U)\} \qquad \ldots (2.2) \end{aligned}$$

Now using the condition (1.5) in (2.2), we get

$$\begin{aligned} & 6'(X, Y, Z, U) + {}'K(Z, Y, X, U) + {}'K(X, Z, Y, U) + {}'K(X, \bar{Z}, Y, \bar{U}) + {}'K(\bar{Z}, Y, X, \bar{U}) \\ & = -2K\{2'F(X, Y)'F(Z, U) + G(X, Z)\, G(Y, U) - G(Y, Z)\, G(X, U) + {}'F(Y, Z)\, 'F(U, X) \\ & + {}'F(X, Z)\, 'F(Y, U)\} \end{aligned}$$

Further using Binchies first identity and (1.5) in this equation, we get

$$\begin{aligned} 4'K(X, Y, Z, U) = & -k\{2'F(X, Y)'F(Z, U) + G(X, Z)\, G(Y, U) \\ & - G(Y, Z)\, G(X, U) - {}'F(Y, Z) + {}'F(X, U) + {}'F(X, Z)\, 'F(Y, U)\} \qquad \ldots (2.3)(a) \end{aligned}$$

$$4K(X, Y, Z) = -k\{2'F(X, Y)\, \bar{Z} + G(X, Z)\, Y - G(Y, Z)\, X - F(Y, Z)\, \bar{X} + {}'F(X, Z)\, \bar{Y}\} \qquad \ldots (2.3)(b)$$

Contracting (2.3) (b) with respect to X, we get

$$4\, \mathrm{Ric}(Y, Z) = k(n+2)\, G(Y, Z) \qquad \ldots (2.4)(a)$$

Again Barring Y, Z in (2.3) (b) and then contracting with respect to X, we get

$$4\, \mathrm{Ric}(\bar{Y}, \bar{Z}) = k(n+2)\, G(Y, Z) \qquad \ldots (2.4)(b)$$

Combining the above two results, we can write

$$\mathrm{Ric}(Y, Z) = \mathrm{Ric}(\bar{Y}, \bar{Z}) = \frac{k(n+2)}{4} G(Y, Z) \qquad \ldots (2.4)(c)$$

Thus, we have

Theorem (2.1). The condition that the para Kähler manifold be of constant holomorphic sectional curvature is given by (2.3) (a) or (2.3) (b).

Corollary (2.1). In a para Kähler manifold with constant holomorphic sectional curvature, we have

$$\text{Ric}\,(Y, Z) = \text{Ric}\,(\bar{Y}, \bar{Z})$$

The proof is obvious from (2.4)(c).

Corollary (2.2). Para kähler manifold with constant holomorphic sectional Curvature is an Einstein manifold.

The proof follows from (2.4)(c).

Now, a Riemanion manifold is defined to be semi-symmetric and Ricci-symmetric [2], [10] if

$$(K, 'K)\,(Z, U, V, W; X, Y) = -\,'K\,(K\,(X, Y, Z), U, V, W) - {}'K\,(Z, K\,(X, Y, U), V, W - {}'K\,(Z, U, K\,(X, Y, V), W) - {}'K\,(Z, U, V, K\,(X, Y, W)) \quad \ldots (2.5)(a)$$

and $$(K.\,\text{Ric})\,(Z, U; X, Y) = -\,\text{Ric}\,(K\,(X, Y, Z), U) - \text{Ric}\,(Z, K\,(X, Y, U)) = 0 \quad \ldots (2.5)(b)$$

Here the tensor $(K'K)$ and $(K.\ \text{Ric})$ are tensor of type (0, 6) and (0, 4). Another tensor Q of type (0, 6) is defined as

$$Q\,(\text{Ric}, 'K)\,(Z, U, V, W; X, Y) = -\,'K\,(\text{Ric}\,(Z, Y)\,X - \text{Ric}\,(Z, X)\,Y, U, V, W) - {}'K\,(Z, \text{Ric}\,(U, Y)\,X - \text{Ric}\,(U, X)\,Y, V, W) - {}'K\,(Z, U, \text{Ric}\,(V, Y)\,X - \text{Ric}\,(V, X)\,Y, W) - {}'K\,(Z, U, V, \text{Ric}\,(W, Y)\,X - \text{Ric}\,(W, X)\,Y) \quad \ldots (2.6)$$

Now, taking account of (2.3), we can write the equation (2.5)(a) as

$$4^2(K, 'K)\,(Z, U, V, W; X, Y) = -\,4'K\,(4K\,(X, Y, Z), U, V, W) - 4'K\,(Z, 4K\,(X, Y, U), V, W) - 4'K\,(Z, U, 4K\,(X, Y, V), W) - 4'K\,(Z, U, V, 4K\,(X, Y, W))$$

Using (2.3) in this equation, we get

$$4^2(K, 'K)\,(Z, U, V, W; X, Y) = -\,4'K\,(-\,k\,\{2'f\,(X, Y)\,\bar{Z} + G\,(X, Z)\,Y - G\,(Y, Z)\,X - {}'F\,(Y, Z)\,\bar{X} + {}'F\,(X, Z)\,\bar{Y}\}, U, V, W) - 4'K\,(Z, -\,k\,\{2'F\,(X, Y)\,\bar{U} + G\,(X, U)\,Y - G\,(Y, U)\,X - {}'F\,(Y, U)\,\bar{X} + ,F\,(X, U)\,\bar{Y}\}, V, W) - 4'K\,(Z, U, -\,k\,\{2'F\,(X, Y)\,\bar{V} + G\,(X, V)\,Y - G\,(Y, V)\,X - {}'F\,(Y, V)\,\bar{X} + {}'F\,(X, V)\,\bar{Y}\}, W) - 4'K\,(Z, U, V, -\,k\,\{2'F\,(X, Y)\,W + G\,(X, W)\,Y - G\,(Y, W)\,X - {}'F\,(Y, W)\,\bar{X} + {}'F\,(X, W)\,\bar{Y}\})$$

$$= 4k\,\{2'F\,(X, Y)\,'K\,(\bar{Z}, U, V, W) + G\,(X, Z)\,'K\,(Y, U, V, W) - G\,(Y, Z)\,'K\,(X, U, V, W)\} - {}'F\,(Y, Z)\,'K\,(\bar{X}, U, V, W) + {}'F\,(X, Z)\,'K\,(\bar{Y}, U, V, W) + 2'F\,(X, Y)\,'K\,(Z, \bar{U}, V, W) + G\,(X, U)'\,K\,(Z, Y, V, W) - G\,(Y, U)\,'K\,(Z, X, V, W) - {}'F\,(Y, U)\,'K\,(Z, \bar{X}, V, W) + {}'F\,(X, U)\,'K\,(Z, \bar{Y}, V, W) + 2'F\,(X, Y)\,'K\,(Z, U, \bar{V}, W) + G\,(X, V)\,'K\,(Z, U, Y, W) - G\,(Y, V)\,'K\,(Z, U, X, W) - {}'F\,(Y, U)\,'K\,(Z, U, \bar{X}, W) + {}'F\,(X, V)\,'K\,(Z, U, \bar{Y}, W) + 2'F\,(X, Y)\,'K\,(Z, U, V, \bar{W}) + G\,(X, W)\,'K\,(Z, U, V, \bar{Y})$$

$$= 4^2\,(K.\,\text{Ric})\,(Z, U; X, Y) - G\,(Y, W)\,'K\,(Z, U, V, X) - {}'F\,(Y, W)\,'K\,(Z, U, V, \bar{X}) + {}'F\,(X, W)'\,K\,(Z, U, V, \bar{Y})$$

Again using (2.3)(a) in the above expression and taking account of the equation (1.1) and (1.2), we obtain

$$4^2\,(K.\,'K)\,(Z, U, Y, W; X, Y) = 0$$

or $$(K.\ 'K)\,(Z, U, Y, W; X, Y) = 0 \qquad \ldots (2.7)$$

Thus, we have

Theorem (2.2). A Para Kähler manifold with constant holomorphic sectional Curvature is Semi-Symmetric

Again using (2.3)(b) in (2.5)(b), we get

$$4^2\ (K.\ \mathrm{Ric})\,(Z, U, X, Y) = -4\ \mathrm{Ric}\ \{4K(X, Y, Z)\ U\} - 4\mathrm{Ric}\ \{Z, 4K(X, Y, U)\}$$
$$= -4\mathrm{Ric}\ (-k\ \{2'F(X, Y)\ \bar{Z} + G(X, Z)\ Y - G(Y, Z)\ X - 'F(Y, Z)\ \bar{X}$$
$$+ 'F(X, Z)\ \bar{Y}, U\}) - 4\mathrm{Ric}\ (Z, -k\ \{2'F(X, Y)\ \bar{U}\} + G(X, U)\ X)$$
$$- 'F(Y, U)\ \bar{X} + 'F(X, U)\ \bar{Y}\})$$

Or

$$= 4k\ [2\ 'F(X, Y)\ \mathrm{Ric}\ (\bar{Z}, U) + G(X, Z)\ \mathrm{Ric}\ (Y, U) - G(Y, Z)\ \mathrm{Ric}\ (X, U)$$
$$- 'F(Y, Z)\ \mathrm{Ric}\ (\bar{X}, U) + 'F(X, Z)\ \mathrm{Ric}\ (\bar{Y}, U) + 2'F(X, Y)\ \mathrm{Ric}\ (Z, \bar{U}) + G(X, U)\ \mathrm{Ric}\ (Z, Y)$$
$$- G(Y, U)\ \mathrm{Ric}\ (Z, Y) - G(Y, U)\ \mathrm{Ric}\ (Z, X) - 'F(Y, U)\ \mathrm{Ric}\ (Z, \bar{X}) + 'F(X, U)\ \mathrm{Ric}\ (Z, \bar{Y})]$$

Now using (2.4)(c) in the above expression, we easily get

$$(K.\ \mathrm{Ric})\,(Z, U; X, Y) = 0$$

or $$(K.\ \mathrm{Ric})\,(Z, U; X, Y) = 0 \qquad \ldots (2.8)$$

Thus, we have

Theorem (2.3). Para Kähler manifold with constant holomorphic sectional Curvature is Ricci Semi –Symmetric.

Further, using (2.3) and (2.4)(c) in (2.6), we have

$$4^2 Q\,(\mathrm{Ric}, 'K)\,(Z, U, V, W; X, Y) = -4^2[\mathrm{Ric}\ (Y, Z)\ 'K(X, U, V, W)$$
$$- \mathrm{Ric}\ (X, Z)\ 'K(Y, U, V, W) + \mathrm{Ric}\ (Y, U)\ 'K(Z, X, V, W)$$
$$- \mathrm{Ric}\ (X, U)\ 'K(Z, Y, V, W) + \mathrm{Ric}\ (Y, V)\ 'K(Z, U, X, W)$$
$$- \mathrm{Ric}\ (X, V)\ 'K(Z, U, Y, W) + \mathrm{Ric}\ (Y, W)\ 'K(Z, U, V, X)$$
$$- \mathrm{Ric}\ (X, W)\ 'K(Z, U, V, Y)]$$

$$= k^2(n+2)\ [G(Y, Z)\ \{2'F(X, U)\ 'F(V, W) + G(X, V)\ G(U, W) - G(U, V)\ G(X, W)$$
$$- 'F(U, V)\ 'F(X, W) + 'F(X, V)\ 'F(U, W)\} - G(X, Z)\ \{2'F(Y, U)\ 'F(V, W)$$
$$+ G(Y, V)\ G(U, W) + G(U, V)\ G(Y, W) - 'F(U, V)\ 'F(Y, W) + 'F(Y, V)\ 'F(U, W)\}$$
$$+ G(Y, U)\ \{2'F(Z, X)\ 'F(V, W) + G(Z, V)\ G(X, W) - G(X, V)\ G(Z, W)$$
$$- 'F(X, V)\ 'F(Z, W) + 'F(Z, V)\ 'F(X, W)\} - G(X, U)\ \{2'F(Z, Y)\ 'F(V, W)$$
$$+ G(Z, V)\ G(Y, W) - G(Y, V)\ G(Z, W) - 'F(Y, V)\ 'F(Z, W) + 'F(Z, V)\ 'F(Y, W)\}$$
$$+ G(Y, V)\ \{2'F(Z, U)\ 'F(X, W) + G(Z, X)\ G(U, W) - G(V, X)\ G(Z, W)$$
$$- 'F(U, X)\ 'F(Z, W) + 'F(Z, X)\ 'F(U, W)\} - G(X, V)\ \{2'F(Z, U)\ 'F(Y, W)$$
$$+ G(Z, Y)\ G(U, W) - G(U, Y)\ G(Z, W) - 'F(U, Y)\ 'F(Z, W) + 'F(Z, Y)\ 'F(U, W)\}$$
$$- G(Y, W)\ \{2\ 'F(Z, U)\ 'F(V, X) + G(Z, V)\ G(U, X) - G(U, V)\ G(Z, X)$$
$$- 'F(U, V)\ 'F(Z, X) + 'F(Z, V)\ 'F(U, X)\} - G(X, W)\ \{2'F(Z, U)\ 'F(V, Y)$$
$$+ G(Z, V)\ G(U, Y) - G(U, Y)\ G(Z, Y) - 'F(U, V)\ 'F(Z, Y) + 'F(Z, V)\ 'F(U, Y)\}$$

$$
\begin{aligned}
4^2 Q\,(\text{Ric}\ \text{‘}K)\,(Z, U, V, W; X, Y) = k^2\,(n+2)\,[2\text{’}F\,(V, W)\,\{&\text{‘}F\,(X, U)\,G\,(Y, Z) \\
&- \text{‘}F\,(Y, U)\,G\,(X, Z) + \text{’}F\,(Y, Z)\,G\,(X, U) - \text{‘}F\,(X, Z)\,G\,(X, U)\}] \\
&+ \text{’}F\,(U, V)\,\{\text{‘}F\,(X, Z)\,G\,(Y, W) - \text{‘}F\,(Y, Z)\,G\,(X, W) \\
&- \text{‘}F\,(X, W)\,G\,(Y, Z) + \text{’}F\,(Y, W)\,G\,(X, Z)\} \\
&+ \text{’}F\,(U, W)\,\{\text{‘}F\,(X, V)\,G\,(Y, Z) - \text{‘}F\,(Y, V)\,G\,(X, Z) \\
&+ \text{’}F\,(Y, Z)\,G\,(X, V) - \text{‘}F\,(X, Z)\,G\,(Y, V)\} \\
&+ \text{’}F\,(Z, W)\,\{\text{‘}F\,(Y, V)\,G\,(X, U) - \text{‘}F\,(X, V)\,G\,(Y, U) \\
&+ \text{’}F\,(X, U)\,G\,(Y, V) - \text{‘}F\,(Y, U)\,G\,(X, V)\} \\
&+ \text{’}F\,(Z, V)\,\{\text{‘}F\,(X, W)\,G\,(Y, U) - \text{‘}F\,(Y, W)\,G\,(X, U) \\
&+ \text{’}F\,(Y, U)\,G\,(X, W) - \text{‘}F\,(X, U)\,G\,(Y, W)\} \\
&+ 2\text{’}F\,(Z, U)\,\{\text{‘}F\,(V, X)\,G\,(Y, W) - \text{‘}F\,(V, Y)\,G\,(X, W) \\
&+ \text{’}F\,(X, W)\,G\,(Y, V) - \text{‘}F\,(Y, W)\,G\,(X, V)\} \qquad \dots (2.8)
\end{aligned}
$$

Interchanging (Z, V) and (U, W) in the above equation, we get

$$Q\,(\text{Ric}, \text{’}K)\,(Z, U, V, W; X, Y) = Q\,(\text{Ric}, \text{’}F)\,(V, W, Z, U; X, Y) \qquad \dots (2.9)$$

So, we have

Theorem (2.4). In a Para Kähler manifold with constant holomorphic sectional Curvature, we have (2.9).

Conclusion

In this paper we corelate Para Kähler manifold with constant holomorphic sectional curvature and here we find that this is Einstein Manifold and semi-Symmetric as well as Ricci –Semi-Symmetric.

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