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ON HYPERBOLIC COSYMPLECTIC MANIFOLD ADMITTING A STRUCTURE CONNECTION–I

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We consider the structure connection B in Hyperbolic Contact Metric manifold and we stuudy (0, 6)-type tensor ($K.'K$) and (0, 4)-type tensor (K. Ric) and Ricci-semii-symmetric, if ($K.'K$) = 0 and (K. Ric) = 0, respectively. Here we obtained the properties of these (0, 6) and (0, 4)-type tensors in a hyperbolic cosymplectic manifold admitting the structure connection. Further, we have shown the necessary and sufficient condition that the hyperbolic Cosymplectic manifold admitting an F-T–structure connection is flat iff the Curvature tensor with respect to B vanishes.

KEYWORDS : Hyperbolic contact metric manifold, Hyperbolic cosymplectic Manifold, Structure connection, Curvature tensor, (0, 6)-type tensor ($K.'K$), (0, 4)-type tensor (K. Ric), Ricci-semi –symmetric manifold, etc.

INTRODUCTION

Upadhyay, M.D., Dubey, K.K. [1] have studied Almost Hyperbolic contact Structure. Kalpana and Srivastava [4], Sinha, B.B. and Yadav, S.L. [2] Doğan, S. doand Karadoğ [8] and Chinca, O., Gonzalec, C. [6] have studied the Structure connection B in Hyperbolic Contact Metric Manifold. Defever, F. and Others [3] have defined a (0, 6)-type tensor ($K.'K$) and (0, 4)-type tensor (K. Ric) and called the manifold Semi-symmetric and Ricci-semi – symmetric if ($K.'K$) = 0 and (K. Ric) = 0 respectively. Sharma, R., Koufogiorgos [5] have studied the locally symmetric and Ricci symmetric contact metric manifold. While Ahmad, M., Ali, K. [7] have studied Semi invariant submanifold of Nearly hyperbolic Cosymplectic Manifold.

Let us consider an odd-dimensional complete real Differentiable manifold M_n of dimension n; with a fundamental tensor field F of type (1, 1), a fundamental vector field T and a 1-form A, satisfying [1]

$$F^2X = X + A(X)\,T \quad \ldots (1.1)\,(a)$$

$$FT = 0 \quad \ldots (1.1)\,(b)$$

$$A(FX) = 0 \quad (1.1)\,(c)$$

$$A(T) = -1, \text{ and rank } (F) = n \quad \ldots (1.1)\,(d)$$

For every tangent vector field X in M_n. then M_n is called a hyperbolic contact metric manifold if a pseudo Riemannian metric tensor g, satisfies

$$G(FX, FY) = -g(X, Y) - A(X)A(Y) \quad \ldots (1.2)(a)$$

$$g(T, X) = A(X) \quad \ldots (1.2)(b)$$

The structure Bundle $\{F, T, A, G\}$ is called hyperbolic contact metric structure [1].

The fundamental 2-form $'F$ of the structure defined as

$$'F(X, Y) = g(FX, Y) \quad \ldots (1.3)(a)$$

Satisfies

$$'F(X, Y) = -\,'F(Y, X) = -\,'F(\bar{X}, \bar{Y})$$

where $$\bar{X} \overset{\text{def}}{=} FX \quad \ldots (1.3)(c)$$

A hyperbolic metric manifold M_n is said to be the hyperbolic cosymplectic

Manifold [1] & [2], if

$$(D_X F)(Y) = 0 \quad \ldots (1.4)(a)$$

$$(D_X A)(Y) = 0 \quad \ldots (1.4)(b)$$

and $$D_X T = 0 \quad \ldots (1.4)(c)$$

where D is the Riemannian connection in M_n. Defever, F. and others (1997) have defined [3].

$$(K.'K)(Z, U, V, W; X, Y) = -\,'K(K(X, Y, Z), U, V, W) - \,'K(Z, K(X, Y, U), V, W) - \,'K(Z, U, K(X, Y, V), W) - \,'K(Z, U.V, K(X, Y, W)) \quad \ldots (1.5)(a)$$

$$(K.\text{ Ric})(Z, U; X, Y) = -\text{Ric}(K(X, Y, Z), U) - \text{Ric}(Z, K(X, Y, U)) \quad \ldots (1.5)(b)$$

where $(K.'K)$ is a tensor of type (0, 6) and $(K.$ Ric$)$ is a tensor of type (0, 4) . Here K and Ric are the Riemannian curvature tensor and Ricci- tensor in M_n.

If $(K.'K) = 0$ and $(K.$ Ric$) = 0$, then the manifold is to be semi-symmetris and Ricci-semi-symmetric, respectively [3] .

Remark (1.1) : In the above and in what follows, X, Y, Z, …….. are the tangent vector fields in M_n.

A STRUCTURE CONNECTION IN HYPERBOLIC COSYMPLECTIC MANIFOLD M_n

The structure connection B in a hyperbolic contact metric manifold M_n is given by [1], [2], [4].

$$B_X Y = D_X Y + H(X, Y) \quad \ldots (2.1)(a)$$

Where $$H(X, Y) = A(Y)X - A(Y)\bar{X} - A(X)\bar{Y} - g(X, Y)T + \,'F(X, Y)T \quad \ldots (2.1)(b)$$

Whose torsion tensor is defined as

$$S^*(X, Y) = A(Y)X - A(X)Y + 2'F(X, Y)T \quad \ldots (2.2)$$

In a hyperbolic contact metric manifold M_n, we have

$$(B_X A)(Y) = (D_X A)(Y) - A(X)A(Y) - g(X, Y) + \,'F(X, Y) \quad \ldots (2.3)(a)$$

$$B_X T = D_X T - A(X)T - X - \bar{X} \quad \ldots (2.3)(b)$$

and $$(B_X F)(Y) = (D_X F)(Y) - A(Y)\bar{X} + A(Y)X - g(X, Y)T - \,'F(X, Y)T \quad \ldots (2.4)$$

Now, we rewrite the equation (2.1) in the form

$$B_Y Z = D_Y Z + A(Z)\,Y - g(Y, Z)\,T + {}'F(Y, Z)\,T - A(Z)\,\bar{Y} - A(Y)\,\bar{Z} \qquad \ldots (2.5)$$

The curvature tensor of the structure connection B in the hyperbolic contact metric manifold M_n is given by

$$R(X, Y, Z) = B_X B_Y Z - B_Y B_X Z - B(X, Y)\,Z \qquad \ldots (2.6)$$

In view of the equation (2.5), we obtain

$$\begin{aligned} R(X, Y, Z) = {} & K(X, Y, Z) + (D_X A)(Z)\,Y - (D_Y A)(Z)\,X - (D_X A)(Z)\,\bar{Y} \\ & + (D_Y A)(Z)\,\bar{X} - g(Y, Z)\,D_X T + g(X, Z)\,D_Y T + {}'F(Y, Z)\,D_X T \\ & - {}'F(X, Z)\,D_Y T - A(Z)\,\{(D_X F)(Y) - (D_Y F)(X)\} + (D_X{}'F)(Y, Z)\,T \\ & - (D_Y{}'F)(X, Z)\,T - A(Y)\,(D_X F)(Z) + A(X)\,(D_Y F)(Z) + g(Y, Z)\,X \\ & - g(X, Z)\,Y - {}'F(Y, Z)\,X + {}'F(X, Z)\,Y - g(Y, Z)\,\bar{X} + g(X, Z)\,\bar{Y} \\ & + {}'F(Y, Z)\,\bar{X} - {}'F(X, Z)\,\bar{Y} - \bar{Z}\,\{(D_X A)(Y) - (D_Y A)(X)\} \qquad \ldots (2.7) \end{aligned}$$

In view of the equation (1.4), the curvature tensor of the structure connection B in hyperbolic cosymplectic manifold M_n is given by

$$R(X, Y, Z) = K(X, Y, Z) + \{g(Y, Z) - {}'F(Y, Z)\}\,(X - \bar{X}) - \{g(X, Z) - {}'F(X, Y)\}\,(Y - \bar{Y}) \qquad \ldots (2.8)$$

Contracting the above equation with respect to X,

We obtain

$$R(Y, Z) = \text{Ric}(Y, Z) + (n-2)\,g(Y, Z) - (n-2)\,{}'F(Y, Z) - A(Y)\,A(Z), \qquad \ldots (2.9)\text{ (a)}$$

or

$$R(Y) = K(Y) + (n-2)\,Y - (n-2)\,\bar{Y} - A(Y)\,T \qquad \ldots (2.9)\text{ (b)}$$

Again contracting (2.9) (b) with respect to Y,

We get

$$r = k + n(n-2) + 1 = k + (n-1)^2 \qquad \ldots (2.10)$$

Here, $R(Y, Z)$ and r are Ricci-tensor and scalar curvature of the structure Connection B, respectively and k is the scalar curvature of manifold M_n.

Let us suppose that the curvaturre tensor of the structure connection vanish, *i.e.*

$$R(X, Y, Z) = 0,$$

Then we have from (2.8), (2.9) and (2.10), the following results :

$$K(X, Y, Z) = \{g(X, Z) - {}'F(X, Z)\}(Y - \bar{Y}) - \{g(Y, Z) - {}'F(Y, Z)\}\,(X - \bar{X}) \qquad \ldots (2.11)$$

$$\text{Ric}(Y, Z) = (n-2)\,{}'F(Y, Z) - (n-2)\,g(Y, Z) + A(Y)\,A(Z) \qquad \ldots (2.12)$$

and

$$k = -(n-1)^2 \qquad \ldots (2.13)$$

From (2.12), we also have

$$\text{Ric}(\bar{Y}, Z) + \text{Ric}(Y, \bar{Z}) = 0 \qquad \ldots (2.14)$$

Thus, we have

Theorem (2.1) : Let M_n be a hyperbolic cosymplectic manifold admitting a structure connection B given by (2.5). If the curvature tensor with respect to B vanishes, then M_n is of constant scalar curvature and

$$\text{Ric}(\bar{Y}, Z) + \text{Ric}(Y, \bar{Z}) = 0$$

Also hold good in M_n.

Proof : Taking $R(X, Y, Z) = 0$ in (2.8) and following the above patterns, we obtain the equation (2.13) and (2.14), from which , the proof of the Theorem follows.

Now, taking account of the equation (1.5)(a),

We write a tensor of type (0, 6),

$$(R.'R)(Z, U, V, W; X, Y) = -\,'R(R(X, Y, Z), U, V, W) - {}'R(Z, R(X, Y.U), V, W) - {}'R(Z, U, R(X, Y, V), W) - {}'R(Z, U, V, R(X, Y, W)) \quad \ldots (2.15)$$

Using (2.8) in the above equation and taking account of the equation (1.2) and (1.5) (a), we obtain, after a long computation,

$$\begin{aligned}(R.'R)&(Z, U, V, W; X, Y) = (K.'K)(Z, U, V, W; X, Y)\\
&- \{g(Y, Z) - {}'F(Y, Z)\}\,[{}'K(X, U, V, W) - {}'K(\bar{X}, U, V, W) + 2\{g(X, W)\\
&- {}'F(X, W)\}\,\{g(U, V) - {}'F(U, V)\} + A(X)A(W)\{g(U, V) - {}'F(U, V)\} - 2\,\{g(X, V)\\
&- {}'F(X, V)\}\,\{g(U, W) - {}'F(U, W)\} - A(X)A(V)\{g(U, W) - {}'F(U, W)\}] + \{g(X, Z)\\
&- {}'F(X, Z)\}\,[{}'K(Y, U, V, W) - {}'K(\bar{Y}, U, V, W) + 2\,\{g(Y, W) - {}'F(Y, W)\}\,\{g(U, V)\\
&- {}'F(U, V)\} + A(Y)A(W)\{g(U, V) - {}'F(U, V)\} - 2\,\{g(Y, V) - {}'F(Y, V)\}\,\{g(U, W)\\
&- {}'F(U, W)\} - A(Y)A(V)\{g(U, W) - {}'F(U, W)\}] - \{g(Y, U) - {}'F(Y, U)\}\,[{}'K(Z, X, V, W)\\
&- {}'K(Z, \bar{X}, V, W) + 2\{g(X, V) - {}'F(X, V)\}\,\{g(Z, W) - {}'F(Z, W)\} + A(X)A(V)\{g(Z, W)\\
&- {}'F(Z, W)\} - 2\,\{g(X, W) - {}'F(X, W)\}\,\{g(Z, V) - {}'F(Z, V)\} - A(X)A(W)\{g(Z, V)\\
&- {}'F(Z, V)\}] + \{g(X, U) - {}'F(X, U)\}\,[{}'K(Z, Y, V, W) - {}'K(Z, \bar{Y}, V, W) + 2\,\{g(Y, V)\\
&- {}'F(Y, V)\}\,\{g(Z, W) - {}'F(Z, W)\} + A(Y)A(V)\{g(Z, W) - {}'F(Z, W) - 2\,\{g(Y, W)\\
&- {}'F(Y, W)\}\,\{g(Z, V) - {}'F(Z, V)\} - A(Y)A(W)\{g(Z, V) - {}'F(Z, V)\}] - \{g(Y, V)\\
&- {}'F(Y, V)\}\,[{}'K(Z, U, X, W) - {}'K(Z, U, \bar{X}, W) + A(X)A(Z)\{g(U, W) - {}'F(U, W)\}\\
&- A(X)A(U)\{g(Z, W) - {}'F(Z, W)\}] + \{g(X, V) - {}'F(X, V)\}\,[{}'K(Z, U, Y, W)\\
&- {}'K(Z, U, \bar{Y}, W) + A(Y)A(Z)\{g(U, W) - {}'F(U, W)\} - A(Y)A(U)\{g(Z, W)\\
&- {}'F(Z, W)\}] - \{g(Y, W) - {}'F(Y, W)\}\,[{}'K(Z, U, V, X) - {}'K(Z, U, V, \bar{X})\\
&+ A(X)A(U)\{g(Z, V) - {}'F(Z, V)\} - A(X)A(Z)\{g(U, V) - {}'F(U, V)\}] + \{g(X, W)\\
&- {}'F(X, W)\}\,[{}'K(Z, U, V, Y) - {}'K(Z, U, V, \bar{Y}) + A(Y)A(U)\{g(Z, V)\\
&- {}'F(Z, V)\} - A(Y)A(Z)\{g(U, V) - {}'F(U, V)\}] \qquad \ldots (2.16)\end{aligned}$$

In a hyperbolic cosymplectic manifold,

We have the following results : —

$$K(X, Y, T) = 0 \qquad \ldots (2.17)\ (a)$$

$${}'K(X, Y, Z, T) = A(K(X, Y, Z)) = 0 \qquad \ldots (2.17)\ (b)$$

$$K(X, Y, \bar{Z}) = \bar{K}(X, Y, Z) \qquad \ldots (2.17)\ (c)$$

$$K(X, Y, \bar{Z}, U) = g(K(X, Y, \bar{Z}), U) = g(\bar{K}(X, Y, Z), U) = g(K(X, Y, Z), \bar{U}) = -\,{}'K(X, Y, Z, \bar{U}) \qquad \ldots (2.17)\ (d)$$

Now putting $V = T$ and $W = T$ in (2.16) and using (2.17) and (1.1)(d), we easily obtain

$$(R.'R)(Z, U, T, T; X, Y) = (K.'K)(Z, U, T, T; X, Y) \qquad \ldots (2.18)\ (a)$$

Again putting $Z = T$ and $U = T$ in (2.16) and using (2.17) and (1.1)(d), we get

$$(R.'R)\,(T, T, V, W; X, Y) = (K.'K)\,(T, T, V, W; X, Y) \qquad \dots (2.18)\,(b)$$

Thus we have,

Theorem (2.2) In a hyperbolic cosymplectic manifold M_n, equipped with a structure connection B given by (2.5), we have

$$(R.'R)\,(Z, U, T, T; X, Y) = (K.'K)\,(Z, U, T, T; X, Y)$$

and

$$(R.'R)\,(T, T, V, W; X, Y) = (K.'K)\,(T, T, V, W; X, Y)$$

Proof : The proof of the theorem is an immediate consequence of the equation (2.18) (a) and (2.18)(b).

Now, we suppose that $R\,(X, Y, Z) = 0$, then we have

Corollary (2.1) : In a hyperbolic cosymplectic manifold M_n, equipped with a structure connection B given by (2.5), whose curvature tensor vanishes, we have

$$(K.'K)\,(Z, U, T, T; X, Y) = 0 \qquad \dots (2.19)\,(a)$$

$$(K.'K)\,(T, T, V, W; X, Y) = 0 \qquad \dots (2.19)\,(b)$$

Proof: -- The proof of the corollary immediately follows from the theorem (2.2)

For $R\,(X, Y, Z) = 0$

Now, taking account of the equation (1.5)(b) and using (2.11) and (2.12) in it, we obtain

$$\begin{aligned}(K.\mathrm{Ric})\,(Z, U; X, Y) &= 2\,(n-2)\,[\{g\,(X, Z) - 'F\,(X, Z)\}\ \{g\,(Y, U) - 'F\,(Y, U)\} \\ &- \{g\,(Y, Z) - 'F\,(Y, Z)\}\ \{g\,(X, U) - 'F\,(X, U)\}] - (n-3)\,[A\,(X)\,A\,(U)\ \{g\,(Y, Z) \\ &- 'F\,(Y, Z)\} - A\,(Y)\,A\,(U)\ \{g\,(X, Z) - 'F\,(X, Z)\}] + (n-1)\,[A\,(X)\,A\,(Z)\ \{g\,(Y, U) \\ &- 'F\,(Y, U)\} - A\,(Y)\,A\,(Z)\ \{g\,(X, U) - 'F\,(X, U)\}] \qquad \dots (2.20)\end{aligned}$$

From which, in view of (1.1)(c), we get

$$\begin{aligned}(K.\mathrm{Ric})\,(Z, U, \bar{X}, \bar{Y}) &= 2\,(n-2)\,[\{'F\,(X, Z) + g\,(\bar{X}, \bar{Z})\}\ \{'F\,(Y, U) + g\,(\bar{Y}, \bar{U})\} \\ &- \{'F\,(Y, Z) + g\,(\bar{Y}, \bar{Z})\}\ \{'F\,(X, U) + g\,(\bar{X}, \bar{U})\}] \qquad \dots (2.21)\end{aligned}$$

Putting $U = T$ and $Z = T$ separately in the above equation, we easily obtain

$$(K.\mathrm{Ric})\,(Z, T; \bar{X}, \bar{Y}) = 0 \qquad \dots (2.22)\,(a)$$

$$(K.\mathrm{Ric})\,(T, U; \bar{X}, \bar{Y}) = 0 \qquad \dots (2.22)\,(b)$$

Thus, we have

Theorem (2.3). A hyperbolic cosymplectic manifold M_n, admitting a structure connection B given by (2.5), whose curvature tensor vanishes, is Ricci-semi-symmetric with respect tu the vector fields $(Z, T; \bar{X}, \bar{Y})$ or $(T, U; \bar{X}, \bar{Y})$.

Proof : The proof of the theorem follows, immediately, from theresults (2.22)(a) and (2.22)(b), for R(X,Y,Z) = 0. Further, we suppose that the structure connection B given by (2.5) is a F-T –Connection, then we must have

$$(B_x F)\,(Y) = 0 \qquad \dots (2.23)\,(a)$$

$$(B_X A)\,(Y) = 0 \qquad \dots (2.23)\,(b)$$

$$B_X T = 0 \qquad \dots (2.23)\,(c)$$

Now, in a hyperbolic cosymplectic manifold, the equation (2.23) (b) and (2.23) (c) give

$$X - \bar{X} = -A\,(X)\,T \qquad \dots (2.24)\,(a)$$

or

$$g\,(X, Y) - 'F\,(X, Y) = -A\,(X)\,A\,(Y) \qquad \dots (2.24)\,(b)$$

Using (2.24)(b) in the equation (2.8), we have

$$R(X, Y, Z) = K(X, Y, Z) \quad \ldots (2.25)$$

Thus, we have

Theorem (2.4). Let M_n be a hyperbolic cosymplectic manifold admitting an F-T structure connection B given by (2.5). Then the curvature tensor of B vanishes iff the manifold is flat.

The proof is obvious, in view of the equation the equation (2.25).

Conclusion

In this paper we coorelate Structure connection B in Hyperbolic Contact Metric Manifold with properties of (0, 6)-type and (0, 4)-type tensor in Hyeprbolic Cosymplectic Manifold and we find the necessary and sufficient condition that Hyperbolic Cosymplectic Manifold admitting an F-T Structure connection is flat, iff the Curvature tensor with respect to B vanishes.

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