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A DISCOURSE ON COUNTABLY gb-COMPACT, SEQUENTIALLY gb-COMPACT, gb-LINDELOF & SECOND COUNTABLE gb-SPACES

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This paper is devoted to introduce and study countably *gb*-compact, sequentially *gb*-compact, *gb*-Lindelof and second countable *gb*-spaces and their inter relationship. In this context the concept of second countable gb-space is projected and inter related to *gb*-Lindelöf space with proper examples.

The *gb*-convergence of a sequence due to regular *b*-open sets in topological space has been conceptualized and the relation of *gb*-convergence with *gb*-continuity and *gb*-irresolute mapping has been discovered here. It also deals with the relation between *gb*-convergent sequence and convergence of a sequence in a space with suitable example.

KEYWORDS : *gb*-continuity, *gb*-convergent sequence, sequentially *gb*-compact space, countably *gb*-compact space, *gb*-lindelöf space, second countable *gb*-space.

INTRODUCTION AND PRELIMINARY

The notions of *b*-open sets and regular *b*-closed sets have been introduced and investigated by D. Andrijevic [1] and N. Nagaveni and A. Narmadha [2] and[3], respectively. In 2007, M. Caldas and S. Jafari projected some applications of *b*-open sets in topological spaces [4] whereas 2009 was the year for the conceptualization of the class of generalized *b*-closed sets and its fundamental properties by A. Al-Omari and M.S.M. Noorami [5].

The class of generalized closed sets and regular generalized closed sets was coined and framed by N. Levine [6] and N. Palanniappan and K. Chandrasekhar Rao [7], respectively.

We, here, introduce and study *gb*-Lindelöf space, countably *gb*-compact space and sequentially *gb*-compact space. We also study the new concept of second countable *gb*-space along with the *gb*-converge of a sequence and its behavior under *gb*-continuity/irresolute in a topological space.

As usual throughout this paper (X, T) means a topological spaces on which no separation axioms are assumed unless otherwise mentioned.

For a subset A of a space (X, T), cl (A) and int (A) stand as the closure of A and the interior of A, respectively.

Also, X-A or A^C represents the complements of A in X.

Now, the following definitions are recalled which are useful in the sequel :

Definition (1.1) : A subset A of a space (X, T) is said to be b-open [1] if

$$A \subset \text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A)).$$

Definition (1.2) : A subset A of a space (X, T) is said to be regular closed [8] if $A = \text{cl}(\text{int}(A))$.

Definition (1.3) : Generalized b-closed (briefly gb-closed) [5] set if $b\text{cl}(A) \subset U$ whenever $A \subset U$ and U is open in X.

Definition (1.4): A subset A of a space (X, T) is said to be

(1) generalized closed (briefly g-closed) [6] set if

$$\text{cl}(A) \subset U \text{ whenever } A \subset U \text{ and } U \text{ is open in } X.$$

(2) generalized semi-closed (briefly gs-closed) [9] set if

$$\text{scl}(A) \subset U \text{ whenever } A \subset U \text{ and } U \text{ is open in } X.$$

(3) semi-generalized closed (briefly sg-closed) [10] set if

$$\text{scl}(A) \subset U \text{ whenever } A \subset U \text{ and } U \text{ is semi-open in } X.$$

(4) regular generalized closed (briefly rg-closed) [7] set if

$$\text{cl}(A) \subset U \text{ whenever } A \subset U \text{ and } U \text{ is regular open in } X.$$

(5) generalized pre-closed (briefly gp-closed) [11] set if

$$\text{pcl}(A) \subset U \text{ whenever } A \subset U \text{ and } U \text{ is open in } X.$$

(6) A subset A of a space (X, T) is said to be generalized b-closed(briefly gb-closed) [3] if

$$\text{rcl}(A) \subset U \text{ whenever } A \subset U \text{ and } U \text{ is } b\text{-open in } (X, T).$$

The compliments of the above mentioned closed sets are their respective open sets.

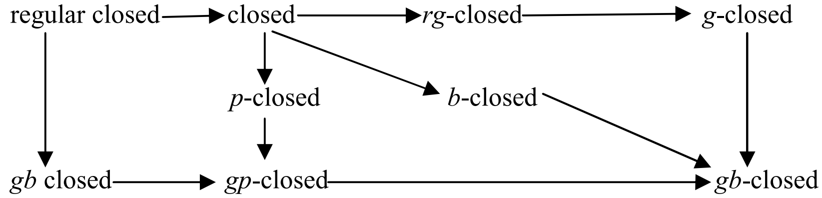
The intersection of all generalized b-closed sets of X containing A is called generalized b-closure of A and is denoted by gb-cl (A).

The union of all regular-open sets of X contained in A is called the generalized b-interior of A and is denoted by gb-int (A).

The family of all gb-open (respectively gb-closed) sets of (X, T) is denoted by $GBO(X)$ (respectively $GBC(X)$). The family of gb-open sets of (X, T) containing a point $x \in X$ is denoted by $GBO(X, x)$.

The following diagram is obtained as a part of diagram in [14].

Diagram



Now, the compactness is dealt with covering the sets by *gb*-open sets as mentioned in the following definitions:

Definition (1.5) : In a topological space (X, T), a collection C of *gb*-open sets in X is called a *gb*-open cover of $A \subseteq X$ if $A \subseteq \cup \{V_r : V_r \in C\}$.

Definition (1.6) : A topological space (X, T) is called a *gb*-compact space/*gb*-Lindelöf space if every cover of X by *gb*-open sets has a finite subcover/countable subcover.

Definition (1.7) : In a topological space (X, T), a subset A of X is said to be *gb*-compact relative to X if for every *gb*-open cover C of A, there is a finite sub collection C^* of C that covers A.

Definition (1.8): A subspace of a topological space, which is *gb*-compact as a topological space in its own right, is said to be *gb*-compact subspace.

The following lemma (1.1) is enunciated for the above definitions to be consistent :

Lemma (1.1) :

(1) Every *gb*-compact space is a *gb*-Lindelöf space.

(2) Every *gb*-Lindelöf space is a Lindelöf space.

(3) Every countable space is a *gb*-Lindelöf space.

(3(a)) A *gb*-Lindelöf space need not be a *gb*-compact space.

(4) *gb*-compactness is not hereditary.

Proof: The **s**tatement follows from definitions (1.6), (1.7) and (1.8).

SECOND COUNTABLE *gb*-SPACE

Definition (2.1) : A topological space (X, T) is said to be a second countable *gb*-space or a second axiom *gb*-space if it carries the following axiom, known as the "Second Axiom of *gb*-countability" (framed analogous to second Axiom of countability) :

[C_1] There exists a countable *gb*-open base for the topology T.

We, however, coin *gb*-open base for the space (X, T) as a sub collection $B \subseteq GBO(X)$ such that every member of T is a union of members of B.

Thus, a topological space (X, T) is called a second countable *gb*-space iff there exists a countable *gb*-open base for T.

Theorem (2.1) : Every second countable *gb*-space is a *gb*-Lindelöf space.

Proof : Let the topological space (X, T) be a second countable *gb*-space.

Let $\{G_b\}_{b\in\Delta}$ be a *gb*-open cover of X, Then

$$X = \bigcup_{b\in\Delta} G_b \qquad \ldots(1)$$

As X being second countable *gb*-space, there exists a countable *gb*-open base for the topology T. Let $B = (V_n)$ be a countable *gb*-open base for T. from (1) it follows that for each $x \in X$, there exists $b_x \in \Delta$ such that

$$x \in G_{b_x} \qquad \ldots(2)$$

Now, since B is a *gb*-open base for T, each open set is a union of some members of B. It, therefore, follows from statement (2) that for each $x \in X$, $\exists V_{n_x} \in B$. Such that

$$x \in V_{n_x} \subseteq G_{r_x} \qquad \ldots(3)$$

Hence, $$X = \bigcup_{x \in X} V_{n_x} \qquad \ldots(4)$$

Since, the family $\{V_{n_x} : x \in X\} \subseteq B$ and B is countable, it follows that the family $\{V_{n_x} : x \in X\}$ is countable. Hence, we can write

$$\{V_{n_x} : x \in X\} = \{V_{n_k} : k \in \Delta_0\} \qquad \ldots(5)$$

where Δ_0 is a countable index set.

This means that for each $k \in \Delta_0, \exists x_k \in X$ such that $V_{n_k} = V_{n_{x_k}}$.

Hence, according to (2) and (3), for each $k \in \Delta_0$, we select one index $\alpha_{x_k} \in \Delta$ such that

$$V_{n_{x_k}} \subseteq G_{r_{x_k}} \qquad \ldots(6)$$

Thus, from (4), (5), (6), we have

$$X = \bigcup_{x \in X} V_{n_x} = \bigcup_{k \in \Delta_0} V_{n_{x_k}} \subseteq \bigcup_{k \in \Delta_0} G_{\alpha_{x_k}}$$

But always $$\bigcup_{k \in \Delta_0} G_{b_{x_k}} \subseteq X.$$

Hence, $$X = \bigcup_{k \in \Delta_0} G_{b_{x_k}} \qquad \ldots(7)$$

Moreover the family $\{G_{r_{x_k}} : k \in \Delta_0\}$ is countable, hence by (7), this family is a countable *gb*-open subcovering of X.

Thus, every second countable *gb*-space is a *gb*-Lindelöf space.

Hence, the theorem.

SEQUENTIALLY gb-COMPACT SPACES

The notion of convergence is fundamental in analysis and topology. Before we take up the concept of sequentially *gb*-compact spaces and countably *gb*-compact spaces, we project the notion of *gb*-convergence of a sequence, *gb*-limit of a sequence, *gb*-accumulation point of a set in a topological space in the following manner:

Definition (3.1) : Let (X, T) be a topological space and $A \subseteq X$.

A point $p \in X$ is called a *gb*-limit point (or a *gb*-cluster point or a *gb*-accumulation point) of A iff every *gb*-open set containing p contains a point of A other than p.

i.e. symbolically $[p \in (X, T) \wedge A \subseteq X] \Rightarrow [p = \text{A } gb\text{-limit point for } A]$

$$\Leftrightarrow [\forall N \in GBO(X) \wedge p \in X \Rightarrow [N-\{p\}] \cap A \neq \varphi]]$$

Definition (3.2) : *gb*-convergent sequences :

A sequence $\{x_n\}$ in a topological space (X, T) is said to be *gb*-convergent to a point x_0 or to converge to a point $x_0 \in X$ with respect to *gb*-open sets, written as $x_n \xrightarrow{gb-cgt} x_0$, if for every *gb*-open set L containing x_0, there exists a positive integer m, s.t. $n \geq m \Rightarrow x_n \in L$.

This concept is symbolically presented as:

$$x_n \xrightarrow{gb-cgt} x_0 \Leftrightarrow gb \lim_{n \to \infty} x_n = x_0$$

Obviously, a sequence $\{x_n\}$ in a topological space (X, T) is said to be *gb*-convergent to a point x_0 in X iff it is eventually in every *gb*-open set containing x_0.

Definition (3.3) : *gb*-limit point of a sequence :

A point x_0 in X is said to be *gb*-limit point of a sequence $\{x_n\}$ in a topological space (X, T) iff every *gb*-open set L containing x_0 there exists a +ve integer n for each +ve integer m such that $n \geq m \Longrightarrow x_n \in L$.

This means that a sequence $\{x_n\}$ in a topological space (X, T) is said to have $x_0 \in X$ as a *gb*-limit point iff for every *gb*-open set containing x_0 contains x_n for finitely many n.

Definition (3.4) : Sequentially *gb*-compact spaces :

A topological space (X, T) is said to be sequentially *gb*-compact iff every sequence in X contains a sub-sequence which is *gb*-convergent to a point of X.

Definition (3.5) : Countably *gb*-compact spaces :

A topological space (X, T) is said to be countably *gb*-compact (or to have *gb*-Bolzano Weierstrass Property) iff every infinite subset of X has at least one *gb*-limit point in X.

Or

A topological space (X, T) is known as countably *gb*-compact iff every countable *T*-*gb*-open cover of X has a finite sub-cover.

Remark (3.1) :

(i) Every finite subspace of a topological space is sequentially *gb*-compact.

(ii) Every *gb*-compact space is a countably *gb*-compact space.

(iii) Every cofinite topological space is a countably *gb*-compact space.

Theorem (3.1) : Every sequentially *gb*-compact topological space (X, T) is countably *gb*-compact .

Proof : Let (X, T) be a sequentially *gb*-compact topological space. Let E be any infinite subset of X. Then there exists an infinite sequence $\{x_n\}$ in E with distinct terms.

Since (X, T) is sequentially *gb*-compact, the sequence $\{x_n\}$ contains a sub sequence $\{x_{nk}\}$ which is *gb*-convergent to $x_0 \in X$.

This means that each *gb*-open set containing x_0 contains an infinite number of elements of E.

Hence, x_0 is an *gb*-accumulation point of E.

Thus, every infinite subset E of X has at least one *gb*-accumulation point in X. Consequently (X, T) is countably *gb*-compact.

i.e. sequentially *gb*-compactness implies countable *gb*-compactness.

Hence, the theorem.

Remark (3.2):

A countably *gb*-compact space is not necessarily sequentially *gb*-compact as illustrated by following example:

Example (3.1):

Let $N = \{n : n \text{ is a natural number}\}$.

Let T be topology on N generated by the family $H = \{\{2n-1, 2n\} : n \in N\}$ of subsets of N.

Let E be a non-empty subset of N.

Let $m_0 \in E$. If m_0 is even $m_0 - 1$ is a *gb*-accumulation point of E and if m_0 is odd $m_0 + 1$ is a *gb*-accumulation point of E. Hence, every non-empty subset of N has a *gb*-accumulation point, so that (N, T) is countably *gb*-compact.

Also, (N, T) is not sequentially *gb*-compact because the sequence $\{2n-1 : n \in N\}$ has no *gb*-convergent sub-sequence.

Therefore,

Countably *gb*-compactness $\not\Rightarrow$ *gb*-sequentially compactness.

$\not\Rightarrow$ *gb*-compactness.

Definition (3.6) : *gb*-continuity at a point :

A mapping $f : (X, T) \to (Y, \sigma)$ from one topological space (X, T) to another topological space (Y, σ) is said to be *gb*-continuous at a point $x_0 \in X$ if for every σ-open set V containing $f(x_0)$ there exists a *gb*-open set L in (X, T) containing x_0 such that $f(L) \subseteq V$.

Definition (3.6) (a) : *gb*-irresolute at a point :

A mapping $f : (X, T) \to (Y, \sigma)$ from one topological space (X, T) to another topological space (Y, σ) is said to be *gb*-irresolute at a point $x_0 \in X$ if for every *gb*-open set V containing $f(x_0)$ there exists a *gb*-open set L in (X, T) containing x_0 such that $f(L) \subseteq V$.

We, here, produce the following two theorems concerned with *gb*-convergence and convergence of a sequence and its image sequence under *gb*-continuity and *gb*-irresoluteness:

Theorem (3.2) : In a topological space (X, T) if a sequence $\{x_n\}$ is *gb*-convergent to a point $x_0 \in X$, then it is also simply convergent to that point . But the converse may not be true.

Proof : Let K be an open set in a topological space (X, T) containing $x_0 \in X$, then K is also a *gb*-open set.

Now, let $\{x_n\}$ be a *gb*-convergent sequence which *gb*-converges to the point $x_0 \in X$. Then for every *gb*-open set L containing x_0 there exists a +ve integer m such that $x_n \in L$ for all $n \geq m$.

Thus, $x_n \xrightarrow{rb-cgt} x_0 \Leftrightarrow \forall L \in RBO(X)$ and $x_0 \in L$ implies that there exists a positive integer $m > 0$ such that $\forall\, n \geq m \Rightarrow x_n \in L$. This is also true for every open set $K \in T$. Since K is an agbitrary open set containing x_0, hence, $x_n \xrightarrow{cgt} x_0$.

But "the converse is not true" is supported by the following example:

Example (3.2) :

Let $X = \{a, b, c\}$, $T = \{\varphi, \{a, b\}, X\}$.

Then $\{b\}$ is a *gb*-open set but not an open set. Let x_n = a for all n, then $x_n \xrightarrow{cgt} a$, as well as $x_n \xrightarrow{cgt} b$, because open subsets containing a and b are $\{a, b\}$ and X.

But $\{x_n\}$ is not *gb-cgt* to "b" because there exists a *gb*-open set containing "b" as $\{b\}$ which does not contain "a".

Hence, the theorem.

Theorem (3.3) : If $f\,(X, T) \longrightarrow (Y, \sigma)$ be a *gb*-continuous mapping from a topological space (X, T) into another topological space (Y, σ) and $\{x_n\}$ be *gb*-convergent to $x_0 \in X$, then $\{f(x_n)\}$ is convergent to $f(x_0) \in Y$.

Proof : Given that the mapping $f : (X, T) \longrightarrow (Y, \sigma)$ is *gb*-continuous so that it is *gb*-continuous at every point of X.

Let $\{x_n\}$ be a sequence in (X, T), which is *gb*-convergent to $x_0 \in X$.

Let V be a σ-open set in (Y, σ) containing $f(x_0)$. Then the *gb*-continuity of f at x_0 implies that there is an *gb*-open set L in (X, T) containing x_0 such that $f(L) \subseteq V$.

Since, $x_n \xrightarrow{rb-cgt} x_0$, there exists an natural number m such that $n \geq m \Longrightarrow x_n \in L \Longrightarrow f(x_n) \in V$. Combining these, we say that the sequence $\{f(x_n)\}$ is *cgt*. to $f(x_0)$ because for every σ-open set V containing $f\,(x_0)$, there exists an natural number m such that $n \geq m \Longrightarrow\ f(x_n) \in V$.

Hence, Symbolically, $x_n \xrightarrow{rb-cgt} x_0 \Rightarrow f(x_n) \xrightarrow{cgt} f(x_0)$, $\forall$ *gb*-continuous maps f.

Hence, the theorem.

Corollary (3.1) : If $f : (X, T) \longrightarrow (Y, \sigma)$ be a *gb*-irresolute mapping and $\{x_n\}$ be *gb*-convergent to $x_0 \in X$, then

$$x_n \xrightarrow{rb-cgt} x_0 \Rightarrow f(x_n) \xrightarrow{rb-cgt} f(x_0).$$

Proof: The proof is straight forward and natural, so omitted.

We, now, produce the following theorem concerned with *gb*-continuous image of a sequentially *gb*- compact set of a topological space.

Theorem (3.4) : A *gb*-continuous image of a sequentially *gb*-compact set is sequentially compact.

Proof : Suppose, f is a *gb*-continuous mapping. Let A be a sequentially *gb*-compact set in topological space (X, T) and we have to show that $f(A)$ is sequentially compact subset of (Y, σ) where $f(X, T) \longrightarrow (Y, \sigma)$.

Let $\{y_n\}$ be an arbitrary sequence of points in $f(A)$, then for each $n \in N$ there exists $x_n \in A$ such that $f(x_n) = y_n$ and thus we obtain a sequence $\{x_n\}$ of points of A.

But A is sequentially *gb*-compact w.r.t. T so that there is a subsequence $\{x_{nk}\}$ of $\{x_n\}$ which is *gb*-compact to a point say, x of A.

Therefore, $x_{nk} \xrightarrow{rb-cgt} x \Rightarrow f(x_{nk}) \rightarrow f(x) \in f(A)$ as f is *gb*-continuous.

Hence, $f(x_{nk})$ is a subsequence of the sequence $\{y_n\}$ of $f(A)$, converging to a point $f(x)$ in $f(A)$. Consequently, $f(A)$ is sequentially compact.

Corollary (3.2) : The *gb*-irresolute image of a sequentially *gb*-compact set is a sequentially *gb*-compact.

This means that sequentially *gb*-compactness is a topological property under *gb*-irresolute mappings.

Conclusion

Since, compactness is one of the most important useful and fundamental concepts in topology so its structural properties as emphasized in the form of *gb*-open sets, *gb*-convergent sequences, *gb*-Lindelöf spaces etc have been analyzed to create the vast canvas in the world of Mathematics through this paper. The structures mentioned in the paper have wide applications and it surely pleases the Mathematician if one of his abstract structures finds an application.

The future scope of study is to obtain results in respective paracompactness.

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