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ON β*g-CLOSED SETS AND β*-NORMAL SPACES

M. C. SHARMA
Department of Mathematics, N. R. E. C. College Khurja-203131 (U.P.), India
AND
HAMANT KUMAR
Department of Mathematics, S. S. (P. G.) College, Shikarpur-203395 (U.P.), India

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In this paper, we introduce the notion of β**g*-closed sets and we show that the family of all β**g*-open sets in a topological space (*X*, τ) is a topology for *X* which is finer than τ. Further we obtain some characterizations and preservation theorems for β*-normality and normality.

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Introduction

The concept of closedness is fundamental with respect to the investigation of topological spaces. Levine [5] initiated the study of the so called *g*-closed set and by doing this he generalized the concept of closedness. β-open sets and β-closed sets were introduced by Monsef *et al*. [1]. Dontchev [2] defined and studied generalized β-closed (briefly *g*β-closed) sets in topological spaces. β-continuity has been introduced by Monsef *et al*. [1]. Mahmoud *et al*. [6] gave the concept of β-irresolute and β-normal spaces in topological spaces. Recently, Sharma and Hamant [8] introduced β-generalized closed (briefly β*g*-closed) sets. In 2011, Thabit and Kamarulhaili [13] presented some characterizations of weakly (resp. almost) regular spaces. Also object of this paper is to present some conditions to assure that the product of two spaces will be π-normal. In 2012, Thabit and Kamarulhaili [14] introduced a weaker version of *p*-normality called π*p*-normality and obtained some basic properties, examples, characterizations and preservation theorems of this property are presented. In 2014, Patil, Benchalli and Gonnagar [9] introduced and studied two new classes of spaces, namely ωα–normal and ωα- ωα-closed sets. In 2015, Hamant *et al* [3] introduce a new class of normal spaces is called π*g*β–normal spaces, by using π*g*β–open sets. We proved that π*g*β–normality is a topological property and it is a hereditary property with respect to π–open, π*g*β-closed subspace. Further we obtain a characterization and preservation theorems for π*g*β-normal spaces.

Preliminaries

Throughout this paper, spaces (X, τ), (Y, σ), and (Z, γ) (or simply X, Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by Cl (A) and Int(A) respectively. A is said to be **β**-open [1] if $A \subset \text{Cl}(\text{Int}(\text{Cl}(A)))$ and preopen [7] (briefly *p*-open) if $A \subset \text{Int}(\text{Cl}(A))$. The family of all β-open (resp. β-closed) sets of X is denoted by $\beta O(X)$ (resp. $\beta C(X)$). The complement of a β-open set is said to be β-closed [1]. The intersection of all β-closed sets containing A is called β-closure of A, and is denoted by βCl (A). The β-Interior of A, denoted by β Int (A), is defined as union of all β-open sets contained in A.

2.1 Definition: A subset A of a space X is said to be

(1) **generalized closed** (briefly ***g*-closed**) [5] if $\text{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

(2) **generalized β-closed** (briefly **gβ-closed**) [2] if $\beta\text{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

(3) **β- generalized closed [8]** (briefly **βg-closed**) if $\beta\text{Cl}(A) \subset U$ whenever $A \subset U$ and U is β-open in X.

The complement of a *g*-closed (resp. *g*β-closed, β*g*-closed) is said to be *g*-open (resp**. *g*β-open, β*g*-open**).

β****g*-CLOSED SETS**

Definition 3.1: A subset A of a space X is said to be

(1) **β****g*-closed** if $\text{Cl}(A) \subset U$ whenever $A \subset U$ and U is β-open in X. The collection of all β**g*-closed subsets in X is denoted by $\beta^*GC(X)$. The intersection of all β**g*-closed sets containing A is denoted by $\beta^*g\text{-Cl}(A)$.

(2) **β****g*-open** if $X \setminus A$ is β**g*-closed. The collection of all β**g*-open subsets in X is denoted by $\beta^*GO(X)$.

Remark 3.2: We have the following implications for the properties of subsets:

$$\begin{array}{ccccc} \text{closed} & \Rightarrow & \beta^*g\text{-closed} & \Rightarrow & g\text{-closed} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \beta\text{-closed} & \Rightarrow & \beta g\text{ closed} & \Rightarrow & g\beta\text{-closed} \end{array}$$

where none of the implications is reversible as can be seen from the following examples:

Example 3.3 : Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a\}, \{a, b\}, X\}$. Then $A = \{b\}$ is *g*β-closed. But it is not *g*-closed not even closed.

Example 3.4: Let $X = \{a, b, c, d\}$ and $\tau = \{\varnothing, \{a, b\}, \{a, b, c\}, X\}$. Then $A = \{a, b, d\}$ is *g*-closed as well as *g*β-closed. But it is not closed.

Example 3.5: Let $X = \{a, b, c,\}$ and $\tau = \{\varnothing, \{a\}, \{b, c\}, \text{X}\}$. Then $A = \{a, c\}$ is β**g*-closed but it is not closed.

Example 3.6: Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a\}, X\}$. Then $A = \{a, b\}$ is β**g*-closed as well as *g*-closed. But it is not β-closed.

Theorem 3.7: The union of two β*g-closed sets (and hence the finite union of β*g-closed sets) in a space X is β*g-closed.

Proof: Let G be a β-open set containing $A \cup B$. Then $\text{Cl}(A) \subset G$ and $\text{Cl}(B) \subset G$ implies that $\text{Cl}(A \cup B) \subset G$. This proves that $A \cup B$ is β*g-closed.

Remark 3.8: Arbitrary union of β*g-closed sets may not be β*g-closed as shown by the following example.

Example 3.9: Let $X = N$ and τ be the cofinite topology. Let $\{A_n : A_n = \{2, 3,\ldots, n+1\}, n \in N\}$ be a collection of β*g-closed sets in X. Then $\cup A_n = N \setminus \{1\} = A$ (say) having a finite complement is open and hence β-open not closed. As $\text{Cl}(A) = N \not\subset A$ gives, A is not β*g-closed.

Definition 3.10 : The intersection of all β–open subsets of a space X containing a set A is called the β-kernel of A and denoted by $\beta \ker (A)$.

Lemma 3.11 : A subset A of a space X is β*g-closed if and only if $\text{Cl}(A) \subset \beta \ker (A)$.

Proof : Assume that A is a β*g-closed set in X. Then $\text{Cl}(A) \subset G$ whenever $A \subset G$ and G is β-open in X. This implies $\text{Cl}(A) \subset \cap \{G : A \subset G \text{ and } G \in \beta O(X)\} = \beta \ker (A)$. For the converse, assume that $\text{Cl}(A) \subset \beta \ker (A)$. This implies $\text{Cl}(A) \subset \cap \{G : A \subset G \text{ and } G \in \beta O(X)\}$. This shows that $\text{Cl}(A) \subset G$ for all β-open sets G containing A. This proves that A is β*g-closed.

Remark 3.12 : Every pre-open set is β-open.

Lemma 3.13 [3, Lemma 2] : Every singleton $\{x\}$ in a space X is either nowhere dense or preopen.

Theorem 3.14 : Arbitrary intersection of β*g-closed sets in a space X is β*g-closed.

Proof : It is obvious.

Corollary 3.15 : For any space (X, τ), $\beta^*GO(X)$ is a topology for X.

β*-NORMAL SPACES

Definition 4.1 : A space X is said to be β-normal [6] (resp. ββ-normal [8]) if for every pair of disjoint closed (resp. β-closed) sets A and B in X, there exist disjoint β-open sets U and V such that $A \subset U$ and $B \subset V$.

Definition 4.2 : A space X is said to be β*-normal if for each pair of disjoint β-closed sets A and B, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

Example 4.3 : Let $X = \{a, b, c, d\}$ and $\tau = \{\varnothing, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$. Then the space X is β-normal but it is not normal.

Example 4.4 : Let $X = \{a, b, c, d\}$ and $\tau = \{\varnothing, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$. Then the space X is ββ–normal as well β-normal. But it is not β*-normal, not even normal.

Remark 4.5 : The following diagram holds. It is shown that normality and β-normality are independent; none of the implications is reversible.

By the definitions and examples stated above, we have the following diagram:

β^*–normality	$\Rightarrow$	$\beta\beta$-normality
$\Downarrow$		$\Downarrow$
normality	$\Rightarrow$	β-normality

Theorem 4.6 : For a topological space X, the following properties are equivalent:

(1) X is β^*-normal;

(2) for any disjoint $H, K \in \beta C(X)$, there exist disjoint β^*g-open sets U, V such that $H \subset U$ and $K \subset V$;

(3) for any $H \in \beta C(X)$ and any $V \in \beta O(X)$ containing H, there exists a β^*g-open set U of X such that $H \subset U \subset \beta^*g\text{-Cl}(U) \subset V$;

(4) for any $H \in \beta C(X)$ and any $V \in \beta O(X)$ containing H, there exists an open set U of X such that $H \subset U \subset \text{Cl}(U) \subset V$;

(5) for any disjoint $H, K \in \beta C(X)$, there exist disjoint regular open sets U, V such that $H \subset U$ and $K \subset V$.

Proof : (1) $\Rightarrow$ (2) : Since every open set is β^*g-open, the proof is obvious.

(2) $\Rightarrow$ (3) : Let $H \in \beta C(X)$ and V be any β-open set containing H. Then $H, X \setminus V \in \beta C(X)$ and $H \cap (X \setminus V) = \varnothing$. By (2), there exist β^*g-open sets U, G such that $H \subset U$, $X \setminus V \subset G$ and $U \cap G = \varnothing$. Therefore, we have $H \subset U \subset X \setminus G \subset V$. Since U is β^*g-open and $X \setminus G$ is β^*g-closed, we obtain $H \subset U \subset \beta^*g\text{-Cl}(U) \subset X \setminus G \subset V$.

(3) $\Rightarrow$ (4) : Let $H \in \beta C(X)$ and $H \subset V \in \beta O(X)$. By (3), there exist a β^*g-open set U_0 of X such that $H \subset U_0 \subset \beta^*g\text{-Cl}(U_0) \subset V$. Since $\beta^*g\text{-Cl}(U_0)$ is β^*g-closed and $V \in \beta O(X)$, $\text{Cl}(\beta^*g\text{-Cl}(U_0)) \subset V$. Put $\text{Int}(U_0) = U$, then U is open and $H \subset U \subset \text{Cl}(U) \subset V$.

(4) $\Rightarrow$ (5) : Let H, K be disjoint β–closed sets of X. Then $H \subset X \setminus K \in \beta O(X)$ and by (4), there exists an open set U_0 such that $H \subset U_0 \subset \text{Cl}(U_0) \subset X \setminus K$. Therefore, $V_0 = X \setminus \text{Cl}(U_0)$ is an open set such that $H \subset U_0$, $K \subset V_0$ and $U_0 \cap V_0 = \varnothing$. Moreover, put $U = \text{Int}(\text{Cl}(U_0))$ and $V = \text{Int}(\text{Cl}(V_0))$, then U, V are regular open sets such that $H \subset U$, $K \subset V$ and $U \cap V = \varnothing$.

(5) $\Rightarrow$ (1) : This is obvious.

By using β^*g-open sets, we obtain a characterization of normal spaces.

Theorem 4.7 : For a topological space X, the following properties are equivalent:

(1) X is normal;

(2) for any disjoint closed sets A and B, there exist disjoint β^*g-open sets U and V such that $A \subset U$ and $B \subset V$;

(3) for any closed set A and any open set V containing A, there exists a β^*g-open set U of X such that $A \subset U \subset \text{Cl}(U) \subset V$.

Proof : (1) $\Rightarrow$ (2) : This is obvious since every open set is β^*g-open.

(2) $\Rightarrow$ (3) : Let A be a closed set and V an open set containing A. Then A and $X \setminus V$ are disjoint closed sets. There exist disjoint β^*g-open sets U and W such that $A \subset U$ and $X \setminus V \subset W$. Since $X \setminus V$ is closed, we have $X \setminus V \subset \text{Int}(W)$ and $U \cap \text{Int}(W) = \varnothing$. Therefore, we obtain $\text{Cl}(U) \cap \text{Int}(W) = \varnothing$ and hence $A \subset U \subset \text{Cl}(U) \subset X \setminus \text{Int}(W) \subset V$.

(3) $\Rightarrow$ (1) : Let A, B be disjoint closed sets of X. Then $A \subset X \setminus B$ and $X \setminus B$ is open. By (3), there exists a β^*g-open set G of X such that $A \subset G \subset \text{Cl}(G) \subset X \setminus B$. Since A is closed, we

have $A \subset \text{Int}(G)$. Put $U = \text{Int}(G)$ and $V = X \setminus \text{Cl}(G)$. Then U and V are disjoint open sets of X such that $A \subset U$ and $B \subset V$. Therefore, X is normal.

FUNCTIONS AND β*-NORMAL SPACES

Definition 5.1 : A function $f : X \to Y$ is said to be :

(1) almost β*g-continuous if for any regular open set V of Y, $f^{-1}(V) \in \beta^*GO(X)$;

(2) almost β*g-closed if for any regular closed set F of X, $f(F) \in \beta^*GC(Y)$.

Definition 5.2 : A function $f : X \to Y$ is said to be :

(1) **β-irresolute** [**6**] (resp. **β-continuous** [1]) if for any β-open (resp. open) set V of Y, $f^{-1}(V)$ is β-open in X;

(2) pre β-closed (resp. **β-closed** [1]) if for any β-closed (resp. closed) set F of X, $f(F)$ is β-closed in Y.

Theorem 5.3 : A function $f : X \to Y$ is an almost β*g-closed surjection if and only if for each subset S of Y and each regular open set U containing $f^{-1}(S)$, there exists a β*g-open set V such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof : Necessity. Suppose that f is almost β*g-closed. Let S be a subset of Y and U a regular open set of X containing $f^{-1}(S)$. Put $V = Y \setminus f(X \setminus U)$, then V is a β*g-open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency : Let F be any regular closed set of X. Then $f^{-1}(Y \setminus f(F)) \subset X \setminus F$ and $X \setminus F$ is regular open. There exists a β*g-open set V of Y such that $Y \setminus f(F) \subset V$ and $f^{-1}(V) \subset X \setminus F$. Therefore, we have $f(F) \supset Y \setminus V$ and $F \subset f^{-1}(Y \setminus V)$. Hence, we obtain $f(F) = Y \setminus V$ and $f(F)$ is β*g-closed in Y. This shows that f is almost β*g-closed.

Theorem 5.4 : If $f : X \to Y$ is an almost β*g-closed β-irresolute (resp. β-continuous) surjection and X is β*-normal, then Y is β*-normal (resp. normal).

Proof : Let A and B be any disjoint β-closed (resp. closed) sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint β-closed sets of X. Since X is β*-normal, there exist disjoint open sets U and V of X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Put $G = \text{Int}(\text{Cl}(U))$ and $H = \text{Int}(\text{Cl}(V))$, then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By Theorem 5.3, there exist β*g-open sets K and L of Y such that $A \subset K$, $B \subset L$. $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, K and L are also disjoint. It follows from Theorem 4.6 (resp. Theorem 4.7) that Y is β*-normal (resp. normal).

Theorem 5.5 : If $f : X \to Y$ is a continuous almost β*g-closed surjection and X is a normal space, then Y is normal.

Proof : The proof is similar to that of Theorem 5.4.

Theorem 5.6 : If $f : X \to Y$ is an almost β*g-continuous pre β-closed (resp. β-closed) injection and Y is β*-normal, then X is β*-normal (resp. normal).

Proof : Let H and K be disjoint β-closed (resp. closed) sets of X. Since f is a pre β-closed (resp. β-closed) injection, $f(H)$ and $f(K)$ are disjoint β-closed sets of Y. Since Y is β*-normal, there exist disjoint open sets P and Q such that $f(H) \subset P$ and $f(K) \subset Q$. Now, put $U = \text{Int}(\text{Cl}(P))$ and $V = \text{Int}(\text{Cl}(Q))$, then U and V are disjoint regular open sets such that

$f(H) \subset U$ and $f(K) \subset V$. Since f is almost β^*g-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint β^*g-open sets such that $H \subset f^{-1}(U)$ and $K \subset f^{-1}(V)$. It follows from Theorem 4.6 (resp. Theorem 4.7) that X is β^*-normal (resp. normal).

Conclusion

I. this paper, we have introduced weak form of normality namely softly-normality and established their relationships with some weak forms of normal spaces in topological spaces.

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