Acta Ciencia Indica, Vol. XLI M, No. 1, 61 (2015)

SAIGO FRACTIONAL INTEGRAL ASSOCIATED WITH WRIGHT'S FUNCTION

VIJAY KUMAR SINGHAL

Swami Keshwanand Institute of Technology, Management and Gramothan, Jaipur-302017, India

RECEIVED : 30 January, 2015

The present paper aims at the study and derivation of Saigo generalized fractional integral operator involving product of Multivariable *H*-function and Wright function. On account of the most general nature of the operator, and the function involved herein, a large number of known and new results involving simpler operators and functions follow as special cases of our main finding. For the sake of illustration some special cases of the main result have been discussed.

KEYWORDS : Siago fractional integral operator, Wright's hypergeometric function, *H*-function of several complex variables, Appel function.

INTRODUCTION

The *H*-function of several complex variables introduced by Srivastava and Panda [20, p. 265] is defined and represented in the following manner:

$$H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left[\begin{matrix}[(a):\theta';\ldots;\theta^{(r)}]:[b':\phi'];\ldots;[b^{(r)}:\phi^{(r)}]\\ [(c):\psi';\ldots;\psi^{(r)}]:[d':\delta'];\ldots;[d^{(r)}:\delta^{(r)}]\end{matrix}\middle| z_1,\ldots,z_r\right]$$

$$=\frac{1}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r} T(s_1,\ldots,s_r)\,R_1(s_1)\cdots R_r(s_r)\,z_1^{s_1}\cdots z_r^{s_r}\,ds_1\cdots ds_r, \quad \ldots(1.1)$$

where $w=\sqrt{(-1)}$,

$$R_i(s_i)=\frac{\prod_{j=1}^{u^{(i)}}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{v^{(i)}}\Gamma(1-b_j^{(i)}+\phi_j^{(i)}s_i)}{\prod_{j=u^{(i)}+1}^{D^{(i)}}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=v^{(i)}+1}^{B^{(i)}}\Gamma(b_j^{(i)}-\phi_j^{(i)}s_i)},\ \forall\ (i=1,2,\ldots,r),$$

$$T(s_1,\ldots,s_r)=\frac{\prod_{j=1}^{\lambda}\Gamma(1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}s_i)}{\prod_{j=\lambda+1}^{A}\Gamma(a_j-\sum_{i=1}^{r}\theta_j^{(i)}s_i)\prod_{j=1}^{C}\Gamma(1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}s_i)},$$

and an empty product is interpreted as unity.

Wright's hypergeometric function [22] is defined by

$$ {}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\ (b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix};z\right]=\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j m)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j m)}\frac{z^m}{m!} $$

where $1+\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j\geq 0,\ \alpha_j\,(j=1,\ldots,p)$ and $\beta_j\,(j=1,\ldots,q)$ are positive real numbers.

The Mellin-Barnes integral of the generalized Wright function [12] is given by

$$ {}_p\Psi_q\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\frac{\Gamma(s)\prod_{i=1}^{p}\Gamma(a_i-\alpha_i s)}{\prod_{j=1}^{q}\Gamma(b_j-\beta_j s)}(-z)^{-s}\,ds \qquad \ldots(1.2) $$

with a special path of integration L.

The Saigo fractional integral operator ([13], [21]) is defined as

$$ I_{0,x}^{p,q,\gamma}f(x)=\begin{cases}\dfrac{x^{-p-q}}{\Gamma(p)}\displaystyle\int_0^x(x-t)^{p-1}F\left(p+q,-\gamma;p;1-\frac{t}{x}\right)f(t)dt & (\mathrm{Re}(p)>0)\\ \dfrac{d^r}{dx^r}I_{0,x}^{p+r,q-r,\gamma-r}f(x), & (\mathrm{Re}(p)\leq 0,\ \ 0<\mathrm{Re}(p)+r\leq 1, r=1,2,\ldots)\end{cases} \qquad \ldots(1.3) $$

where F is the Gauss hypergeometric function.

Saigo fractional integral operator contains as special cases, the Riemann-Liouville and Erdély-Kober operators of Fractional Integration of order $\alpha > 0$ [6] :

$$ I_{0,z}^{\alpha,-\alpha,-\alpha}f(z)=R^{\alpha}f(z)=\frac{z^{\alpha}}{\Gamma(\alpha)}\int_0^1(1-t)^{\alpha-1}f(tz)dt $$

$$ z^{-\alpha-\gamma}I_{0,z}^{\alpha,-\alpha-\gamma,-\alpha}f(z)=I_1^{\gamma,\alpha}f(z)=\frac{1}{\Gamma(\alpha)}\int_0^1(1-t)^{\alpha-1}t^{\gamma}f(zt)dt \qquad (\alpha>0,\ \gamma\in \mathrm{R}) $$

Let $\alpha,\ \alpha',\ \beta,\ \beta' \in R$ and $\gamma > 0$, then Saigo generalized fractional integral operator [13] of a function $f(x)$ is defined by

$$ I_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma}f(z)=\frac{z^{-\alpha}}{\Gamma(\gamma)}\int_0^z(z-t)^{\gamma-1}t^{-\alpha'}F_3\left(\alpha,\ \alpha',\ \beta,\ \beta';\ \gamma;1-\frac{t}{z},\ 1-\frac{z}{t}\right)f(t)\,dt \qquad \gamma>0 \quad \ldots(1.4) $$

where $f(z)$ is analytic in a simply connected region of z-plane. Principal value for $0 \leq \arg(z-t) \leq 2\pi$ is denoted by $(z-t)^{\gamma-1}$

The Appell hypergeometric function of third type, also known as Horn's $\boldsymbol{F_3}$ function is defined as

$$F_3(\alpha,\alpha';\beta,\beta';\gamma;z,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{z^m t^n}{m!n!}\qquad |z|<1,\ |t|<1$$

Following Lemma [13, p. 394]; see also [7] will be required in the sequel:

Lemma : Let $\text{Re}(\gamma)>0, k>\max[0,\text{Re}(\alpha+\alpha'+\beta-\gamma),\text{Re}(\alpha'-\beta')]-1$ then

$$I_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma}[z^k]=\frac{\Gamma(1+k)\Gamma(1+k-\alpha'+\beta')\Gamma(1+k-\alpha-\alpha'-\beta+\gamma)}{\Gamma(1+k+\beta')\Gamma(1+k-\alpha'-\beta+\gamma)\Gamma(1+k-\alpha-\alpha'+\gamma)}z^{k-\alpha-\alpha'+\gamma}\qquad ...(1.5)$$

Main result

$$I_{0,t}^{\alpha,\alpha',\beta,\beta',\gamma}\Bigg[t^{\sigma}(p-qt)^{\rho}\ {}_m\Psi_n\left(t^{\lambda_1}(p-qt)^{-\mu}\right)$$

$$H_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}\left(\begin{matrix}[(a):\theta';\ldots;\theta^{(r)}]:[b':\phi'];\ldots;[b^{(r)}:\phi^{(r)}]\\ [(c):\psi';\ldots;\psi^{(r)}]:[d':\delta'];\ldots;[d^{(r)}:\delta^{(r)}]\end{matrix}\ \middle|\ \begin{matrix}z_1 t^{\delta_1}(p-qt)^{-\eta_1}\\ \vdots\\ z_r t^{\delta_r}(p-qt)^{-\eta_r}\end{matrix}\right)\Bigg]$$

$$=p^{\rho}t^{\sigma-\alpha-\alpha'+\gamma}\ H_{A+4,C+4:[B',D'];\ldots;[B^{(r)},D^{(r)}];(m,n+1);(0,1)}^{0,\lambda+4:(u',v');\ldots;(u^{(r)},v^{(r)});(1,m);(1,0)}\left[\begin{matrix}z_1t^{\eta_1}\\ \vdots\\ z_rt^{\eta_r}\\ -t^{\lambda_1}\\ -t\end{matrix}\ \middle|\ \begin{matrix}(1+\rho;q_1,q_2,\ldots,q_r,\mu,1),\\ (1+\rho;q_1,q_2,\ldots,q_r,\mu,0),\end{matrix}\right.$$

$$(-\sigma;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),(-\sigma+\alpha'-\beta';\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),$$
$$(-\sigma-\beta';\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),(-\sigma+\alpha'+\beta'-\gamma;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),$$
$$(-\sigma+\alpha+\alpha'+\beta-\gamma;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),[(a):\theta';\ldots;\theta^{(r)},0,\ldots,0]:$$
$$(-\sigma+\alpha+\alpha'-\gamma;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),[(c):\psi';\ldots;\psi^{(r)},0,\ldots,0]:$$

$$\left.\begin{matrix}[b':\phi'];\ldots;[b^{(r)}:\phi^{(r)}];[1-a_i:\alpha_i];\ \_\ ;\ \_\\ [d':\delta'];\ldots;[d^{(r)}:\delta^{(r)}];[1-b_j:\beta_j];(0,1),(0,1)\end{matrix}\right]\qquad \ldots(2.1)$$

Provided

(1) $\alpha,\alpha',\beta,\beta',\gamma,\mu,\lambda_1,\sigma,\rho\in C$

(2) $\gamma>0,\text{Re}(\sigma)+\sum_{i'=1}^{r}\delta_i\min_{1\le j\le u^{(r)}}\left[\text{Re}\dfrac{d_j^{(i')}}{\delta_j^{(i')}}\right]+1>\max\{0,\alpha'-\beta',\alpha+\beta-\gamma\}-1$

(3) $\left|\dfrac{q}{p}t\right|<1$

Proof: In order to prove (2.1), firstly the H-function and Wright's function are expressed in terms of Mellin-Barnes type of contour integrals (1.1) and (1.2) respectively, followed by interchanging of the order of summations and integration, which is permissible under the stated conditions. Now, by virtue of the Lemma (1.5), the desired result is readily obtained after a little simplification.

Interesting Special Cases

On account of the most general character of the *H*-function and wright's function occurring in the main result, many special cases of the result can be derived but, for the sake of brevity, a few interesting special cases are recorded here.

(i) Setting $m = 1 = n, a_1 = 1 = \alpha_1, b_1 = B, \beta_1 = A'$ in (2.1), we get

$$I_{0,t}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma}(p-qt)^{\rho}E_{A',B}\left(t^{\lambda_1}(p-qt)^{-\mu}\right)\right.$$

$$\left.H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[(a):\theta';\ldots;\theta^{(r)}]:[b':\phi'];\ldots;[b^{(r)}:\phi^{(r)}]\\ [(c):\psi';\ldots;\psi^{(r)}]:[d':\delta'];\ldots;[d^{(r)}:\delta^{(r)}]\end{matrix}\left|\begin{matrix}z_1 t^{\delta_1}(p-qt)^{-\eta_1}\\ \vdots\\ z_r t^{\delta_r}(p-qt)^{-\eta_r}\end{matrix}\right.\right)\right]$$

$$= p^{\rho}t^{\sigma-\alpha-\alpha'+\gamma}\, H^{0,\lambda+4:(u',v');\ldots;(u^{(r)},v^{(r)});(1,1);(1,0)}_{A+4,C+4:[B',D'];\ldots;[B^{(r)},D^{(r)}];(1,2);(0,1)}\left[\begin{matrix}z_1 t^{\eta_1}\\ \vdots\\ z_r t^{\eta_r}\\ -t^{\lambda_1}\\ -t\end{matrix}\left|\begin{matrix}(1+\rho;q_1,q_2,\ldots,q_r,\mu,1),\\ (1+\rho;q_1,q_2,\ldots,q_r,\mu,0),\end{matrix}\right.\right.$$

$$\begin{matrix}(-\sigma;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),(-\sigma+\alpha'-\beta';\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),\\ (-\sigma-\beta';\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),(-\sigma+\alpha'+\beta'-\gamma;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),\end{matrix}$$

$$\begin{matrix}(-\sigma+\alpha+\alpha'+\beta-\gamma;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),[(a):\theta';\ldots;\theta^{(r)},0,\ldots,0]:\\ (-\sigma+\alpha+\alpha'-\gamma;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),[(c):\psi';\ldots;\psi^{(r)},0,\ldots,0]:\end{matrix}$$

$$\left.\begin{matrix}[b':\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}];[0:1];\ \_\ ;\ \_\\ [d':\delta'];\ldots;[d^{(r)}:\delta^{(r)}];[1-B:A'];(0,1),(0,1)\end{matrix}\right] \qquad \ldots(3.1)$$

valid under the conditions derived from those mentioned for (2.1).

(ii) On taking $\lambda = A = C = 0$, eq. (2.1) reduces to the following form

$$I_{0,t}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma}(p-qt)^{\rho}\,{}_m\Psi_n\left(t^{\lambda_1}(p-qt)^{-\mu}\right)\prod_{i=1}^{r}H^{u^{(i)},v^{(i)}}_{B^{(i)},D^{(i)}}\left(z_i t^{\delta_i}(p-qt)^{-\eta_i}\left|\begin{matrix}[(b^{(i)}):\phi^{(i)}]\\ [(d^{(i)}):\delta^{(i)}]\end{matrix}\right.\right)\right]$$

$$= p^{\rho}t^{\sigma-\alpha-\alpha'+\gamma}\, H^{0,4:(u',v');\ldots;(u^{(r)},v^{(r)});(1,m);(1,0)}_{4,4:[B',D'];\ldots;[B^{(r)},D^{(r)}];(m,n+1);(0,1)}\left[\begin{matrix}z_1 t^{\eta_1}\\ \vdots\\ z_r t^{\eta_r}\\ -t^{\lambda_1}\\ -t\end{matrix}\left|\begin{matrix}(1+\rho;q_1,q_2,\ldots,q_r,\mu,1),\\ (1+\rho;q_1,q_2,\ldots,q_r,\mu,0),\end{matrix}\right.\right.$$

$$\begin{matrix}(-\sigma;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),(-\sigma+\alpha'-\beta';\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),\\ (-\sigma-\beta';\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),(-\sigma+\alpha'+\beta'-\gamma;\eta_1,\eta_2,\ldots,\eta_r,\lambda_1,1),\end{matrix}$$

$$(-\sigma+\alpha+\alpha'+\beta-\gamma;\eta_1,\eta_2,...,\eta_r,\lambda_1,1),-:$$
$$(-\sigma+\alpha+\alpha'-\gamma;\eta_1,\eta_2,...,\eta_r,\lambda_1,1),-:$$

$$\left.\begin{matrix}[b':\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}];[1-a_i:\alpha_i]; & \_ ; & \_ \\ [d':\delta'];\ldots;[d^{(r)}:\delta^{(r)}];[1-b_j:\beta_j];(0,1),(0,1) & & \end{matrix}\right] \quad \ldots(3.2)$$

valid under the conditions derived from those mentioned for (2.1).

Conclusion

On account of being general and unified in nature, the main result of the present paper provides unification and generalization of numerous results lying in the literature. Also, a number of new results in terms of simpler operator and functions can be obtained as special cases of the main result which may find useful applications in the fields of pure and applied mathematics.

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