Acta Ciencia Indica, Vol. XLI M, No. 1, 51 (2015)

EXTENDED FRACTIONAL FOURIER TRANSFORM OF SOME SPECIAL FUNCTIONS

A. S. GUDADHE

Govt. Vidarbha Institute of Science and Humanities, Amravati. (M. S.)

AND

NAVEEN A.

Vidya Bharati Mahavidyalaya, Amravati. (M. S.)

RECEIVED : 24 January, 2015

The extended fractional Fourier transform is the generalization of fractional Fourier transform with two more parameters. In this paper, we have proposed a new definition of extended fractional Fourier transform of a function as an operation which correspond to rotation of Wigner distribution of a function. Moreover we have also obtained extended fractional Fourier transform of some special functions including Gaussian function.

KEYWORDS : Extended fractional Fourier transform (EFrFT), fractional Fourier transform (FrFT), delta function, Gaussian function and Wigner distribution (WD).

INTRODUCTION

As a generalization of the classical Fourier transform (FT), the fractional Fourier transform (FrFT) has received much attention in recent years. Namias [5] had introduced it and since then it has been applied in several areas, including optics, quantum mechanics and signal processing [1, 6, 7], its relationship with the Fourier transform can be found [8].

In 2014, Kai Lu *et. al*. [4] had applied the local fractional *Z*-transforms to signals on cantor sets.

The generalization of the fractional Fourier transform, which is known as extended fractional Fourier transform (EFrFT) introduced by Juanwen Hua *et. al*. [3] (with two more parameters a and b) is,

$$F_{a,b}^{\alpha}[f(t)](u) = F_{a,b}^{\alpha}(u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)\cot\alpha - 2abtucsc\alpha]} f(t)dt \quad \text{... (1.1)}$$

The FrFT of some common functions are seen in [5,7].

If the Gaussian with zero mean and standard deviation σ is,

$$f(t) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{t^2}{2\sigma^2}\right) \quad \text{... (1.2)}$$

Then the fractional Fourier transform of Gaussian is introduced by C. Capus and K. Brown [2] in 2003,

$$F_\alpha(u_\alpha) = \frac{\sqrt{2\pi}A_\alpha}{\sqrt{\left(\frac{1}{\sigma_t^2} - icot\varphi\right)}} \exp\left(\frac{1}{2} iu_\alpha^2 cot\varphi\right) exp\left[-\frac{1}{2}\frac{u_\alpha^2}{\left(\frac{1}{\sigma_t^2} - icot\varphi\right) sin^2\varphi}\right] \quad \text{... (1.3)}$$

where as the Fourier transform of Gaussian function is $F(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\sigma_t^2\right)$.

The relation between Wigner distribution of FrFT of a function $f(u)$ rotated by α is given by Ozaktas *et. al*. [7] as,

$$W_{f_a}(u, \mu) = W_f(u \cos\alpha - \mu \sin\alpha, u \sin\alpha + \mu \cos\alpha) \quad \text{... (1.4)}$$

In this paper we have introduced EFrFT of some special functions and Gaussian function and established a relation in Wigner distribution and EFrFT.

Some special functions of EFrFT

Property 1: If $F_{a,b}^\alpha(u)$ is the EFrFT of $\delta(t)$ then

$$F_{a,b}^\alpha(u) = F_{a,b}^\alpha[\delta(t)](u) = e^{i\pi b^2u^2 \cot\alpha}$$

Proof: By (1.1),

$$F_{a,b}^\alpha[\delta(t)](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)\cot\alpha - 2abtucsc\alpha]}\,\delta(t)dt$$

$$= e^{i\pi b^2u^2\cot\alpha}$$

Property 2 : If $F_{a,b}^\alpha(u)$ is the EFrFT of $\delta(t-\tau)$ then

$$F_{a,b}^\alpha(u) = F_{a,b}^\alpha[\delta(t-\tau)](u) = e^{i\pi[(a^2\tau^2+b^2u^2)\cot\alpha - 2ab\tau ucsc\alpha]}$$

Proof : By (1.1),

$$F_{a,b}^\alpha[\delta(t-\tau)](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)\cot\alpha - 2abtucsc\alpha]}\,\delta(t-\tau)dt$$

By the shifting and definition of EFRFT

$$= e^{i\pi b^2u^2cot\alpha}e^{i\pi a^2(\tau)^2\cot\alpha - 2i\pi ab(\tau)ucsc\alpha}$$

$$= e^{i\pi[(a^2\tau^2+b^2u^2)\cot\alpha - 2ab\tau ucsc\alpha]}$$

Property 3: If $F_{a,b}^\alpha(u)$ is the EFrFT of 1 then

$$F_{a,b}^\alpha(u) = F_{a,b}^\alpha[1](u) = \frac{1}{a}\sqrt{\frac{i}{cot\alpha}}\,e^{-i\pi b^2u^2\tan\alpha}$$

Proof: By (1.1),

$$F_{a,b}^{\alpha}[1](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)\cot\alpha - 2abtucsc\alpha]}\, 1.dt$$

$$= e^{i\pi b^2u^2\cot\alpha}\sqrt{\frac{\pi}{-i\pi a^2\cot\alpha}}\; e^{\frac{-i\pi(abucsc\alpha)^2}{a^2cot\alpha}}$$

$$= \frac{1}{a}\sqrt{\frac{i}{\cot\alpha}}\, e^{-i\pi b^2u^2\tan\alpha}$$

Property 4: If $F_{a,b}^{\alpha}(u)$ is the EFrFT of $e^{2i\pi\xi t}$ then

$$F_{a,b}^{\alpha}(u) = F_{a,b}^{\alpha}\left[e^{2i\pi\xi t}\right](u) = \frac{1}{a}\sqrt{\frac{i}{\cot\alpha}}\, e^{-i\pi b^2u^2\tan\alpha - \frac{i\pi}{a^2}\xi^2\tan\alpha + 2i\pi\frac{b}{a}\xi usec\alpha}$$

Proof: By (1.1),

$$F_{a,b}^{\alpha}\left[e^{2i\pi\xi t}\right](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)\cot\alpha - 2abtucsc\alpha]}\, e^{2i\pi\xi t}dt$$

$$= e^{i\pi b^2u^2\cot\alpha}\sqrt{\frac{\pi}{-i\pi a^2\cot\alpha}}\, e^{\frac{-i\pi(abucsc\alpha-\xi)^2}{a^2\cot\alpha}}$$

$$= \frac{1}{a}\sqrt{\frac{i}{\cot\alpha}}\, e^{-i\pi b^2u^2\tan\alpha - \frac{i\pi}{a^2}\xi^2\tan\alpha + 2i\pi\frac{b}{a}\xi usec\alpha}$$

Property 5: If $F_{a,b}^{\alpha}(u)$ is the EFrFT of $e^{i\pi\chi t^2}$ then

$$F_{a,b}^{\alpha}(u) = F_{a,b}^{\alpha}\left[e^{i\pi\chi t^2}\right](u) = \sqrt{\frac{i\tan\alpha}{(a^2+\chi\tan\alpha)}}\, e^{i\pi b^2u^2\left(\frac{\chi - a^2\tan\alpha}{a^2+\chi\tan\alpha}\right)}$$

Proof: By (1.1),

$$F_{a,b}^{\alpha}\left[e^{i\pi\chi t^2}\right](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)\cot\alpha - 2abtucsc\alpha]}\, e^{i\pi\chi t^2}dt$$

$$= e^{i\pi b^2u^2\cot\alpha}\sqrt{\frac{\pi}{-i\pi(a^2\cot\alpha+\chi)}}\; e^{\frac{-i\pi(abucsc\alpha)^2}{(a^2cot\alpha+\chi)}}$$

$$= \sqrt{\frac{i\tan\alpha}{(a^2+\chi\tan\alpha)}}\, e^{i\pi b^2u^2\left(\frac{\chi - a^2\tan\alpha}{a^2+\chi\tan\alpha}\right)}$$

Property 6: If $F_{a,b}^{\alpha}(u)$ is the EFrFT of $e^{i\pi(\chi t^2+2\xi t)}$ then

$$F_{a,b}^{\alpha}(u) = F_{a,b}^{\alpha}\left[e^{i\pi(\chi t^2+2\xi t)}\right](u) = \sqrt{\frac{i\tan\alpha}{(a^2+\chi\tan\alpha)}}\, e^{\frac{i\pi[b^2u^2(\chi - a^2\tan\alpha) - \xi^2\tan\alpha + 2ab\xi usec\alpha]}{(a^2+\chi\tan\alpha)}}$$

Proof: By (1.1),

$$F_{a,b}^{\alpha}\left[e^{i\pi(\chi t^2+2\xi t)}\right](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)cot\alpha-2abtucsc\alpha]}\, e^{i\pi(\chi t^2+2\xi t)}dt$$

$$= e^{i\pi b^2u^2\cot\alpha}\sqrt{\frac{\pi}{-i\pi(a^2\cot\alpha+\chi)}}\, e^{\frac{-i\pi(abucsc\alpha-\xi)^2}{(a^2\cot\alpha+\chi)}}$$

$$= \sqrt{\frac{i\tan\alpha}{(a^2+\chi\tan\alpha)}}\, e^{\frac{i\pi[b^2u^2(\chi-a^2\tan\alpha)-\xi^2\tan\alpha+2ab\xi usec\alpha]}{(a^2+\chi\tan\alpha)}}$$

Property 7: If $F_{a,b}^{\alpha}(u)$ is the EFrFT of $e^{-\pi t^2}$ then

$$F_{a,b}^{\alpha}(u) = F_{a,b}^{\alpha}\left[e^{-\pi t^2}\right](u) = \sqrt{\frac{1}{1-ia^2\cot\alpha}}\, e^{-\pi b^2u^2\left(\frac{\cot\alpha+ia^2}{a^2\cot\alpha+i}\right)}$$

Proof: By (1.1),

$$F_{a,b}^{\alpha}\left[e^{-\pi t^2}\right](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)\cot\alpha-2abtucsc\alpha]}\, e^{-\pi t^2}dt$$

$$= e^{i\pi b^2u^2\cot\alpha}\sqrt{\frac{\pi}{-i\pi(a^2\cot\alpha+i)}}\, e^{\frac{-i\pi(abucsc\alpha)^2}{(a^2\cot\alpha+i)}}$$

$$= \sqrt{\frac{1}{1-ia^2\cot\alpha}}\, e^{-\pi b^2u^2\left(\frac{\cot\alpha+ia^2}{a^2\cot\alpha+i}\right)}$$

Property 8: If $F_{a,b}^{\alpha}(u)$ is the EFrFT of $e^{-\pi\chi t^2}$ then

$$F_{a,b}^{\alpha}(u) = F_{a,b}^{\alpha}\left[e^{-\pi\chi t^2}\right](u) = \sqrt{\frac{1}{\chi-ia^2\cot\alpha}}\, e^{\frac{i\pi b^2u^2\cot\alpha(\chi^2-a^4)}{a^4\cot^2\alpha+\chi^2}}\, e^{-\frac{\pi a^2b^2u^2\chi csc^2\alpha}{a^4\cot^2\alpha+\chi^2}}$$

Proof: By (1.1),

$$F_{a,b}^{\alpha}\left[e^{-\pi\chi t^2}\right](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)\cot\alpha-2abtucsc\alpha]}\, e^{-\pi\chi t^2}dt$$

$$= e^{i\pi b^2u^2\cot\alpha}\sqrt{\frac{\pi}{-i\pi(a^2\cot\alpha+i\chi)}}\, e^{\frac{-i\pi(abucsc\alpha)^2}{(a^2\cot\alpha+i\chi)}}$$

$$= \sqrt{\frac{1}{\chi-ia^2\cot\alpha}}\, e^{i\pi b^2u^2\left(\frac{i\chi\cot\alpha+a^2\cot^2\alpha-a^2csc^2\alpha}{a^2\cot\alpha+i\chi}\right)}$$

$$= \sqrt{\frac{1}{\chi-ia^2\cot\alpha}}\, e^{\frac{i\pi b^2u^2\cot\alpha(\chi^2-a^4)}{a^4\cot^2\alpha+\chi^2}}\, e^{-\frac{\pi a^2b^2u^2\chi csc^2\alpha}{a^4\cot^2\alpha+\chi^2}}$$

Property 9 : If $F_{a,b}^{\alpha}(u)$ is the EFrFT of $e^{-\pi(\chi t^2+2\xi t)}$ then

$$F_{a,b}^{\alpha}(u) = F_{a,b}^{\alpha}\left[e^{-\pi(\chi t^2+2\xi t)}\right](u)$$

$$= \sqrt{\frac{i}{(\chi - ia^2\cot\alpha)}}\, e^{i\pi\cot\alpha\left(\frac{b^2u^2(\chi^2-a^4)+a^2\xi^2+2ab\xi\chi usec\alpha}{a^4\cot^2\alpha+\chi^2}\right)} e^{-\pi csc^2\alpha\left(\frac{a^2b^2\chi u^2-\chi\xi^2\sin^2\alpha+2a^3b\xi ucos\alpha}{a^4\cot^2\alpha+\chi^2}\right)}$$

Proof : By (1.1),

$$F_{a,b}^{\alpha}\left[e^{-\pi(\chi t^2+2\xi t)}\right](u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)\cot\alpha-2abtucsc\alpha]}\, e^{-\pi(\chi t^2+2\xi t)}dt$$

$$= e^{i\pi b^2u^2\cot\alpha}\sqrt{\frac{\pi}{-i\pi(a^2\cot\alpha+i\chi)}}\, e^{\frac{-i\pi(abucsc\alpha-i\xi)^2}{a^2\cot\alpha+i\chi}}$$

$$= \sqrt{\frac{i}{(\chi - ia^2\cot\alpha)}}\, e^{\left(\frac{i\pi a^2b^2u^2\cot^2\alpha-\pi b^2u^2\chi\cot\alpha-i\pi a^2b^2u^2csc^2\alpha+i\pi\xi^2-2\pi ab\xi ucsc\alpha}{a^2\cot\alpha+i\chi}\right)}$$

$$= \sqrt{\frac{i}{(\chi - ia^2\cot\alpha)}}\, e^{i\pi\cot\alpha\left(\frac{b^2u^2(\chi^2-a^4)+a^2\xi^2+2ab\xi\chi usec\alpha}{a^4\cot^2\alpha+\chi^2}\right)} e^{-\pi csc^2\alpha\left(\frac{a^2b^2\chi u^2-\chi\xi^2\sin^2\alpha+2a^3b\xi ucos\alpha}{a^4\cot^2\alpha+\chi^2}\right)}$$

Table of some special functions of EFrFT:

S. No.	$f(t)$	Extended fractional Fourier transform
1	$\delta(t)$	$e^{i\pi b^2u^2\cot\alpha}$
2	$\delta(t-\tau)$	$e^{i\pi[(a^2\tau^2+b^2u^2)\cot\alpha-2ab\tau ucsc\alpha]}$
3	1	$\frac{1}{a}\sqrt{\frac{i}{\cot\alpha}}\, e^{-i\pi b^2u^2\tan\alpha}$
4	$e^{2i\pi\xi t}$	$\frac{1}{a}\sqrt{\frac{i}{\cot\alpha}}\, e^{-i\pi b^2u^2\tan\alpha-\frac{i\pi}{a^2}\xi^2\tan\alpha+2i\pi\frac{b}{a}\xi usec\alpha}$
5	$e^{i\pi\chi t^2}$	$\sqrt{\frac{i\tan\alpha}{(a^2+\chi\tan\alpha)}}\, e^{i\pi b^2u^2\left(\frac{\chi-a^2\tan\alpha}{a^2+\chi\tan\alpha}\right)}$
6	$e^{i\pi(\chi t^2+2\xi t)}$	$\sqrt{\frac{i\tan\alpha}{(a^2+\chi\tan\alpha)}}\, e^{\frac{i\pi[b^2u^2(\chi-a^2\tan\alpha)-\xi^2\tan\alpha+2ab\xi usec\alpha]}{(a^2+\chi\tan\alpha)}}$
7	$e^{-\pi t^2}$	$\sqrt{\frac{1}{1-ia^2\cot\alpha}}\, e^{-\pi b^2u^2\left(\frac{\cot\alpha+ia^2}{a^2\cot\alpha+i}\right)}$
8	$e^{-\pi\chi t^2}$	$\sqrt{\frac{1}{\chi-ia^2\cot\alpha}}\, e^{\frac{i\pi b^2u^2\cot\alpha(\chi^2-a^4)}{a^4\cot^2\alpha+\chi^2}}\, e^{-\frac{\pi a^2b^2u^2\chi csc^2\alpha}{a^4\cot^2\alpha+\chi^2}}$

9	$e^{-\pi(\chi t^2+2\xi t)}$	$\sqrt{\frac{i}{(\chi-ia^2\cot\alpha)}}\, e^{i\pi\cot\alpha\left(\frac{b^2u^2(\chi^2-a^4)+a^2\xi^2+2ab\xi\chi u sec\alpha}{a^4\cot^2\alpha+\chi^2}\right)} \times e^{-\pi csc^2\alpha\left(\frac{a^2b^2\chi u^2-\chi\xi^2\sin^2\alpha+2a^3b\xi u cos\alpha}{a^4\cot^2\alpha+\chi^2}\right)}$

Gaussian in extended fractional Fourier transform domain

By (1.1)and (1.2) extended fractional Fourier transform of the Gaussion gives,

$$F_{a,b}^{\alpha}(u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2+b^2u^2)\cot\alpha-2abtucsc\alpha]}\frac{1}{\sqrt{2\pi\sigma^2}}exp\left(-\frac{t^2}{2\sigma^2}\right)dt$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}exp(i\pi b^2u^2\cot\alpha)exp\left(-\frac{2\pi^2a^2b^2u^2csc^2\alpha}{\left(\frac{1}{\sigma^2}-2i\pi a^2\cot\alpha\right)}\right)$$

$$\times \int_{-\infty}^{\infty} exp\left[-\frac{1}{2}\left(\sqrt{\left(\frac{1}{\sigma^2}-2i\pi a^2\cot\alpha\right)}\ t+\frac{2i\pi abucsc\alpha}{\sqrt{\frac{1}{\sigma^2}-2i\pi a^2\cot\alpha}}\right)^2\right]dt \quad ...(3.1)$$

Let

$$z = \sqrt{\left(\frac{1}{\sigma^2}-2i\pi a^2\cot\alpha\right)}\ t+\frac{2i\pi abucsc\alpha}{\sqrt{\frac{1}{\sigma^2}-2i\pi a^2 cot\alpha}} \text{ then } dz = \sqrt{\left(\frac{1}{\sigma^2}-2i\pi a^2\cot\alpha\right)}\ dt$$

Therefore (3.1) will be,

$$F_{a,b}^{\alpha}(u) = \frac{1}{\sqrt{2\pi\sigma^2}\sqrt{\frac{1}{\sigma^2}-2i\pi a^2 cot\alpha}}exp(i\pi b^2u^2cot\alpha).exp\left(-\frac{2\pi^2a^2b^2u^2csc^2\alpha}{\left(\frac{1}{\sigma^2}-2i\pi a^2\cot\alpha\right)}\right)$$

$$\times \int_{-\infty}^{\infty} exp\left(-\frac{z^2}{2}\right)dz \quad ...(3.2)$$

The standard definition of the Gaussian gives,

$$\frac{1}{\sqrt{2\pi}}\int exp\left(-\frac{z^2}{2}\right)dz = 1.0 \quad ...(3.3)$$

Therefore by(3.3), (3.2) gives,

$$F_{a,b}^{\alpha}(u) = \frac{1}{\sqrt{1-2i\pi\sigma^2a^2cot\alpha}}exp(i\pi b^2u^2\cot\alpha).exp\left(-\frac{2\pi^2a^2b^2\sigma^2u^2csc^2\alpha}{(1-2i\pi\sigma^2a^2\cot\alpha)}\right)$$

WIGNER DISTRIBUTION OF EFrFT

The Wigner distribution of a signal $f(t)$ is,

$$W_{F_{a,b}^{\alpha}}(t,u) = \int_{-\infty}^{\infty} F_{a,b}^{\alpha}\left(t+\frac{1}{2}x\right) F_{a,b}^{*\alpha}\left(t-\frac{1}{2}x\right) e^{-2i\pi xu} dx \quad \dots (4.1)$$

Then,

$$W_{f_{a,b}^{\alpha}}\left(\frac{b}{a}t\cos\alpha - \frac{u}{ab}\sin\alpha, abt\sin\alpha + \frac{au}{b}\cos\alpha\right) = \int_{-\infty}^{\infty} f\left(\frac{b}{a}t\cos\alpha - \frac{u}{ab}\sin\alpha + \frac{1}{2}x\right)$$

$$\times f^{*}\left(\frac{b}{a}t\cos\alpha - \frac{u}{ab}\sin\alpha - \frac{1}{2}x\right) e^{-2i\pi x\left(abtsin\alpha + \frac{au}{b}\cos\alpha\right)} dx \quad \dots (4.2)$$

and EFrFTs,

$$F_{a,b}^{\alpha}\left(t+\frac{1}{2}x\right) = \int_{-\infty}^{\infty} K_{b,a}^{\alpha}\left(t+\frac{1}{2}x, y\right) f(y)dy$$

$$= \int_{-\infty}^{\infty} e^{i\pi\left[\left(a^2y^2 + b^2\left(t+\frac{1}{2}x\right)^2\right)\cot\alpha - 2aby\left(t+\frac{1}{2}x\right)csc\alpha\right]} f(y)dy \quad \dots (4.3)$$

$$F_{a,b}^{*\alpha}\left(t-\frac{1}{2}x\right) = \int_{-\infty}^{\infty} K_{b,a}^{*\alpha}\left(t-\frac{1}{2}x, z\right) f^{*}(z)dz$$

$$= \int_{-\infty}^{\infty} e^{-i\pi[(a^2z^2 + b^2\left(t-\frac{1}{2}x\right)^2)cot\alpha - 2abz\left(t-\frac{1}{2}x\right)csc\alpha]} f^{*}(z)dz \quad \dots (4.4)$$

Therefore from (4.3) and (4.4), (4.1) will be,

$$W_{F_{a,b}^{\alpha}}(t,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\pi(a^2y^2 + b^2t^2 + \frac{b^2x^2}{4} + b^2tx)\cot\alpha - 2i\pi abytcsc\alpha - i\pi abyxcsc\alpha}$$

$$\times e^{-i\pi\left(a^2z^2 + b^2t^2 + \frac{b^2x^2}{4} - b^2tx\right)\cot\alpha + 2i\pi abztcsc\alpha - i\pi abzxcsc\alpha} e^{-2i\pi xu} f(y)f^{*}(z)dy.dz.dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2i\pi x(u + ab\frac{(y+z)}{2}csc\alpha - b^2t\cot\alpha)} dx. e^{i\pi a^2(y^2 - z^2)\cot\alpha - 2i\pi abt(y-z)csc\alpha} f(y)f^{*}(z)dy.dz$$

Substituting the delta function,

$$\int_{-\infty}^{\infty} e^{-2i\pi x(u + ab\frac{(y+z)}{2}csc\alpha - b^2t\cot\alpha)} dx = \delta(u + ab\frac{(y+z)}{2}csc\alpha - b^2t\cot\alpha)$$

Then,

$$W_{F_{a,b}^{\alpha}}(t,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(u + ab\frac{(y+z)}{2}csc\alpha - b^2t\cot\alpha)$$

$$\times e^{i\pi a^2(y^2 - z^2)\cot\alpha - 2i\pi abt(y-z)csc\alpha} f(y)f^{*}(z)dy.dz$$

$$= \frac{1}{\left|\frac{ab}{2} csc\alpha\right|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta \left[z - \left(-y - 2\left(\frac{u}{ab} \sin\alpha - \frac{b}{a} t \cos\alpha \right) \right) \right]$$

$$\times e^{i\pi a^2 (y^2 - z^2) \cot\alpha - 2i\pi abt(y-z) csc\alpha} f(y) f^*(z) dy.dz$$

By shifting property of the delta function,

$$\int_{-\infty}^{\infty} \delta \left[z - \left(-y - 2\left(\frac{u}{ab} sin\alpha - \frac{b}{a} tcos\alpha \right) \right) \right] f^*(z) dz = f^* \left(-y - 2\left(\frac{u}{ab} sin\alpha - \frac{b}{a} tcos\alpha \right) \right)$$

Therefore,

$$W_{F_{a,b}^{\alpha}}(t,u) = \left| \frac{2\sin\alpha}{ab} \right|$$

$$\int_{-\infty}^{\infty} e^{i\pi a^2 y^2 \cot\alpha - i\pi a^2 \left(-y - 2\left(\frac{u}{ab} \sin\alpha - \frac{b}{a} t \cos\alpha \right) \right)^2 \cot\alpha - 2i\pi abyt csc\alpha + 2i\pi abt \left(-y - 2\left(\frac{u}{ab} \sin\alpha - \frac{b}{a} t\cos\alpha \right) \right) csc\alpha}$$

$$f^* \left(-y - 2\left(\frac{u}{ab} \sin\alpha - \frac{b}{a} t \cos\alpha \right) \right) f(y) dy$$

$$= \left| \frac{2\sin\alpha}{ab} \right| e^{-4i\pi \left(\frac{u}{b} \sin\alpha - bt\cos\alpha \right)^2 \cot\alpha - 4i\pi bt \left(\frac{u}{b} \sin\alpha - bt\cos\alpha \right) csc\alpha}$$

$$\times \int_{-\infty}^{\infty} e^{-4i\pi ay \left(\frac{u}{b} \cos\alpha + bt\sin\alpha \right)} f^* \left(-y - \frac{2}{a} \left(\frac{u}{b} \sin\alpha - bt\cos\alpha \right) \right) f(y) dy \qquad \dots (4.5)$$

By (4.2), substituting $\frac{b}{a} tcos\alpha - \frac{u}{ab} sin\alpha + \frac{1}{2} x = y$, then

$$x = 2y - 2\left(\frac{b}{a} t\cos\alpha - \frac{u}{ab} \sin\alpha \right) \text{ and } dx = 2dy$$

$$W_{f_{a,b}^{\alpha}} \left(\frac{b}{a} t\cos\alpha - \frac{u}{ab} \sin\alpha, abt\sin\alpha + \frac{au}{b} \cos\alpha \right)$$

$$= 2 \int_{-\infty}^{\infty} f(y) f^* \left(\frac{b}{a} t\cos\alpha - \frac{u}{ab} \sin\alpha - y + \frac{b}{a} t\cos\alpha - \frac{u}{ab} \sin\alpha \right)$$

$$\times e^{-2i\pi \left(2y - 2t\frac{b}{a} \cos\alpha + \frac{2u}{ab} \sin\alpha \right) \left(abt\sin\alpha + \frac{au}{b} \cos\alpha \right)} dy$$

$$= 2e^{-4i\pi \left(\frac{u}{b} \sin\alpha - bt\cos\alpha \right)^2 \cot\alpha - 4i\pi bt \left(\frac{u}{b} \sin\alpha - bt\cos\alpha \right) csc\alpha}$$

$$\times \int_{-\infty}^{\infty} e^{-4i\pi ay \left(\frac{u}{b} \cos\alpha + bt\sin\alpha \right)} f(y) f^* \left(-y - \frac{2}{a} \left(\frac{u}{b} \sin\alpha - bt\cos\alpha \right) \right) dy \qquad \dots (4.6)$$

By (4.5) and (4.6),

$$W_{F_{a,b}^{\alpha}}(t,u) = \left| \frac{\sin\alpha}{ab} \right| W_{f_{a,b}^{\alpha}} \left(\frac{b}{a} t\cos\alpha - \frac{u}{ab} \sin\alpha, abt\sin\alpha + \frac{au}{b} \cos\alpha \right)$$

Thus Wigner Distribution of $F_{a,b}^{\alpha}$ is $\frac{\sin\alpha}{ab}$ times Wigner Distribution of $f(y)$ rotated by an angle α.

If we put $a = b = 1$ then we get special case for FRFT defined by equation (1.4) in FRFT.

Conclusion

In this paper we have obtained extended fractional Fourier transform of some common functions and the Gaussian formula in the extended fractional Fourier transform domain. Then we have established the relation of Wigner distribution and EFrFT.

References

1. Almeida, L. B., The fractional Fourier transform and time-frequency representations, *IEEE Trans. Signal Process.,* Vol. **42**, No. **11**, 3084-3091 Nov. (1994).
2. Capus, C. and Brown, K., Fractional Fourier transform of the Gaussian and fractional domain signal support, *IEE Proc.-Vis. Image Signal Process*., Vol. **150**, No. **2**, 99-106 Apr. (2003).
3. Hua, J., Liren, Liu and Guoqiang, Li, Extended fractional Fourier transforms, *J. Opt. Soc. Am.* A/Vol. **14**, No. **12**, 3316-3322 Dec. (1997).
4. Kai, Lu, Ren-Jie, Hu, Carlo, Cattani, Gong-Nan, Xie, Xiao-Jun, Yang and Yang, Zhao, Local fractional Z-transforms with applications to signals on cantor sets, *Abstract and Applied Analysis*, Vol. **2014**, 1-6 (2014).
5. Namias, V., The fractional order Fourier transform and its application to quantum mechanics, *J. Inst. Math. Appl*., **25**, 241-265 (1980).
6. Ozaktas, H.M. and Mendlovic, D., Fractional Fourier transform and their optical implementation, *I. J. Opt. Soc. Am*., **A 10**,1875-1881 (1993).
7. Ozaktas, H.M., Zulevsky, Z. and Kutay, M.A., *The fractional Fourier transform with Applications in Optics and Signal Process*, John Wiley and Sons Chichester (2001).
8. Zayed, A. I., On the relationship between the Fourier and fractional Fourier transforms, *IEEE Signal Process. Letters*, Vol. **3**, No. **12**, 310-311 Dec. (1996).

❑