Acta Ciencia Indica, Vol. XLI M, No. 1, 47 (2015)

A UNIQUE FIXED POINT THEOREM IN HILBERT SPACE

K. KALYANI

Assistant Professor, Department of Science and Humanities, Vignan's University, Vadlamudi- 522 213, Guntur, India

N. SESHAGIRI RAO

Associate Professor and Head, Department of Humanities and Science, Vignan's Lara Institute of Technology and Science, Vadlamudi – 522 213, Guntur

AND

K.V.L.N. ACHARYULU

Associate Professor, Department of Mathematics, Bapatla Engineering College, Bapatla, Guntur -522 101

RECEIVED : 2 January, 2015

REVISED : 24 February, 2015

In this paper a fixed theorem has been proved for a self mapping involving square terms which generalize the classical Banach's contraction mapping principle in the Hilbert space.

KEYWORDS : Hilbert space, closed subset, Cauchy sequence, completeness.

Mathematics Subject Classification : 40A05, 46E20, 47H10, 54H25.

Introduction

A self mapping f defined on metric space (X, d) is called a contraction map if for some $0 < k < 1$,

$$d(f(x), f(y)) \leq kd(x, y) \text{ , for all } x, y \in X$$

Banach (1922) established the existence of the unique fixed point for a contraction map in a complete metric space. This celebrated principle has been generalized by many authors Chu and Diaz [7], Holmes [9], Reich [15], Hardy and Rogers [8], Wong [17], Smart [16] etc. in taking various mappings on different spaces and in early years. In twenty first century the work in fixed theory has been rapidly expanded for taking weaker conditions on the spaces as well as on the defined mappings on the spaces. Some results also have found by taking the sequence of mappings on the spaces. Branciari [5], Bonsall [6], Kannan [10], Nadlar [12] have investigated results on fixed point theorem for mappings satisfying a general contractive condition of integral type. Later Zhang [18], Agarwal *et al*. [2], Altun and Simsek [3] have discussed fixed theorems for some new generalized contractive type mappings and in partially ordered metric spaces. Common fixed point of four maps in partially ordered metric spaces is discussed by Abbas *et al*. [1], Amini-Harandi *et al*. [4]. Results on weakly contractive maps studied Radenovic *et al*. [14]. Recently a common fixed point result for weakly increasing mappings satisfying generalized contractive type of Zhang [18] in ordered metric spaces are derived by Nashine and Altun [13]. Kalyani *et al*. [11] have studied a unique fixed point on Hilbert space with rational term in the inequality.

The main aim of this paper is to find a fixed point of a self mapping T on a closed subset X of a Hilbert space H satisfying inequality having square terms. The theorem follows with the statement

Theorem : Let X be a closed subset of a Hilbert space H and T be a self mapping defined on X satisfying

$$\|Tx - Ty\|^2 \le a\|x - y\|^2 + b\|x - Ty\|^2 + c\|y - Tx\|^2$$

for all $x, y \in X$ and $x \ne y$, where $a, b, c > 0$ with $0 \le a + b + 4c < 1$. Then T has a unique fixed point in X.

Proof : Let us consider $x_0 \in X$. Then we define a sequence $\{x_n\}$ of iterations T of as follows:

$$x_1 = Tx_0,\ x_2 = Tx_1,\ x_3 = Tx_2,\ \ldots\ldots i.e.\ x_{n+1} = Tx_n, \text{ for } n = 0, 1, 2, 3, \ldots\ldots$$

Case (i) : For some n, $x_{n+1} = x_n$, then it immediately follows that x_n is a fixed point of T.

Case (ii) : Now we suppose that $x_{n+1} \ne x_n$, for every $n = 0, 1, 2, 3, \ldots\ldots$

Then, we have

$$\|x_{n+1} - x_n\|^2 = \|Tx_n - Tx_{n-1}\|^2, \quad n \ge 1$$

Now by making use of the hypothesis, we get

$$\|x_{n+1} - x_n\|^2 \le a\|x_n - x_{n-1}\|^2 + b\|x_n - Tx_{n-1}\|^2 + c\|x_{n-1} - Tx_n\|^2$$

$$\Rightarrow \quad (1 - 2c)\|x_{n+1} - x_n\|^2 \le (a + 2c)\|x_n - x_{n-1}\|^2$$

$$\therefore \quad \|x_{n+1} - x_n\| \le k\|x_n - x_{n-1}\|, \quad \text{where } k = \left(\frac{a + 2c}{1 - 2c}\right)^{\frac{1}{2}}$$

Clearly $k < 1$, as $0 < a + b + 4c < 1$. Using the inequality from the hypothesis successively, we get

$$\|x_{n+1} - x_n\|^2 \le k^n \|x_1 - x_0\|$$

$$\therefore \quad \|x_{n+1} - x_n\| \to 0 \quad \text{as } n \to \infty$$

Now, we have to show that $\{x_n\}$ is a Cauchy sequence in X. For this, for every positive integer p, we have

$$\|x_n - x_{n+p}\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \ldots\ldots\ldots\ldots\ldots + \|x_{n+p-1} - x_{n+p}\|$$

$$\le \frac{k^n}{1 - k}\|x_1 - x_0\| \quad \to 0, \quad n \to \infty \text{ because } k < 1$$

Hence $\{x_n\}$ is a Cauchy sequence in X and since X is closed, there exists an element $\mu \in X$ which is the limit of the sequence $\{x_n\}$. Next, we have to show that μ is a fixed point of T. In view of the hypothesis it is observe that

$$\|\mu - T\mu\|^2 \le \|\mu - x_n\|^2 + a\,\|x_{n-1} - \mu\|^2 + b\,\|x_{n-1} - T\mu\|^2 + c\,\|\mu - x_n\|^2 + 2\,\|\mu - x_n\|\ \|x_n - T\mu\|$$

Taking $n \to \infty$, we find that

$$\|\mu - T\mu\|^2 \le b\,\|\mu - T\mu\|^2$$

Since $0 < b < 1$, it follows that $T\mu = \mu$. Hence μ is a fixed point of T. For the uniqueness of the fixed point, let $v, \mu \ne v$ in X be another fixed point of T. Then it is clear that

$$\|\mu - v\|^2 \le a\,\|\mu - v\|^2 + b\,\|\mu - Tv\|^2 + \|v - T\mu\|^2$$

which implies that

$$\|\mu - v\|^2 \le (a + b + c)\,\|\mu - v\|^2$$

This is a contradiction for $a + b + c < 1$. Hence T has a unique fixed point in X.

Conclusion

The result which is found here is the generalization of the Kann's type condition and the result of Koparde and Waghmode. By taking variation in the last two terms of the above mention result,we can get the result of Pandhare and Waghmode.

Acknowledgment

The authors are thankful to the reviewers for their valuable suggestions.

The authors are very much grateful to Prof. N. Ch. Pattabhi Ramacharyulu, Former Faculty, Department of Mathematics, National Institute of Technology, Warangal, India, for his encouragement and valuable suggestions to prepare this article.

References

1. Abbas, M., Nazir, T. and Radenovic, S., Common fixed point of four maps in partially ordered metric spaces, *Appl. Math. Lett.*, No. **9**, 1520-1526 (2011).
2. Agarwal, R. P., El-Gebeily, M. A. and O'Regan, D., Generalized contractions in partially ordered metric spaces, *Appl. Anal*., **87**, No. **1**, 109-116 (2008).
3. Altun, I. and Simsek, H., Some fixed point theorems on ordered metric spaces and application, *Fixed Point Theory and Appl.,* Art. ID **621469**, 17 pp (2010).
4. Amini-Harandi, A. and Emami, H., A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations, *Nonlinear Anal*., **72**, No. **5**, 2238-2242 (2010).
5. Branciari, A., A fixed point theorem for mappings satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci*., **29**, No. **9**, 531-536 (2002).

6. Bonsall, F.F., *Lectures on Some Fixed Point Theorems of Functional Analysis,* Tata Institute of Fundamental Research, Bombay (1962).
7. Chu, S.C., Diaz, J.B., Remarks on generalization of Banach's contraction mapping, *J. Math. Anal. Applic*., **11**, 440-44 (1965).
8. Hardy, G.C. and Rogers, T., A generalization of fixed point of Reich, *Canad. Math. Bull*., **16**, 201-206 (1973).
9. Holmes, R.D., On fixed and periodic points under sets of mappings, *Canad. Math. Bull*., **12**, 813-822 (1969).
10. Kannan, R., Some results on fixed points-II, *Amer. Math. Monthly*, **76**, 405-408 (1969).
11. Kalyani, K., Rao, N. Seshagiri and Acharyulu, K.V.L.N., A fixed point of self mapping in Hilbert space, *Global Journal of Pure and Applied Mathematics*, Vol. **10**, No. **6**, 783-786 (2014).
12. Nadler, S.B., Sequence of contraction and fixed points, *Pacific J. Math*., **27**, 579-585 (1968).
13. Nashine, H. K. and Altun, I., A common fixed point theorem on ordered metric spaces, *Bull. of the Iranian Math. Soc*., Vol. **38**, No. **4**, pp 925-934 (2012).
14. Radenovic, S., Kadelburg, Z., Jandrlic, D. and Jandrlic, A., Some results on weakly contractive maps, *Bull. Iranian Math. Soc*., **38**, No. **3**, 625-645 (2012).
15. Reich, S., Some remarks concerning contraction mappings, *Canad. Math. Bull*., **14**, 121-124 (1971).
16. Smart, D.R., *Fixed point Theorems*, Combridge University Press (1974).
17. Wong, C.S., Common fixed points of two mappings, *Pacific J. Math*., **48**, 299-312 (1973).
18. Zhang, X., Common fixed point theorems for some new generalized contractive type mappings, *J. Math. Anal. Appl*., **333**, No. **2**, 780-786 (2007).

❑