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COMMUTATIVE LATTICE ORDERED GROUP IMPLICATION ALGEBRA

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In this paper, two definitions for commutative lattice ordered group implication algebra (or) commutative *l*-group implication algebra are introduced and it is established that they are equivalent. Some examples of commutative *l*-group implication algebra are given and established that class of commutative *l*-group implication algebras lies between class of *l*-groups and class of Boolean algebras. The relation between commutative *l*-group implication algebra, Browerian algebra, Boolean ring with identity are established and the Characterization for Commutative *l*-group implication algebra is also discussed.

KEYWORDS : Boolean algebra, *l*-group implication algebra, commutative *l*-group implication algebra, Boolean ring with identity and Browerian implication algebra.

INTRODUCTION

It is well known that a distributive complimented lattice is a Boolean algebra which is equivalent to Boolean ring with identity. Conversely Boolean ring with identity is equivalent to Boolean algebra. From this relation class of Lattice ordered groups (or) *l*-groups lies between class of lattices and class of Boolean algebras (rings). Hence Birkhaff, G. posed the problem. "Develop a common abstraction which includes Boolean Algebras (rings) and lattice ordered groups as special cases [problem 115 in [3])". Many common abstractions namely Dually residuated lattice ordered groups or DRl-groups, lattice ordered commutative groups, lattice ordered near rings, lattice ordered modules are presented in [4], [6], [1] and [5] respectively. In connection with above problem we have introduced commutative *l*-group implication algebra in this paper.

PRILIMINARIES

In this section are listed a number of definitions and results which are made use of throughout the paper. The symbols $\leq$, $+$, $-$, $\vee$, $\wedge$, $\rightarrow$, $*$ and $\in$ will denote inclusion, sum, difference, join (least upper bound), meet (greatest lower bound), implication, symmetric difference and membership in a lattice L or commutative l-group implication algebra G. Small letters $a, b, \ldots$ will denote elements of the lattice L or commutative l-group G.

Definition 1.1 : A lattice L is called bounded lattice if it has least element 0 and greatest element 1. A bounded lattice L is called complemented lattice for each a in L there exists a' in L such that $a \vee a' = 1$, $a \wedge a' = 0$.

Boolean Algebra B is a distributive complemented lattice.

Definition 1.2 : A ring R is called Boolean ring if $a^2 = a$ for all a in R. A ring R is called Boolean ring with 1 if there exist $1 \in R$ such that $1.a = a.1 = a$ for all $a \in R$.

Theorem 1.1 : If R is a Boolean ring then

(i) $a + a = 0$ for all $a \in R$.

(ii) $a\,b = b\,a$ for all $a, b \in R$.

Theorem 1.2 : The following systems are equivalent

(i) Boolean ring B with 1

(ii) Boolean algebra B

Definition 1.3 : A non-empty set G is called an l-group if and only if

(i) $(G, +)$ is a group

(ii) $(G, \leq)$ is a lattice

(iii) If $x \leq y$ then $a + x + b \leq a + y + b$ for all a, b, x, y in G.

Or

$$(a + (x \vee y) + b = (a + x + b) \vee (a + y + b)$$

$$a + (x \wedge y) + b = (a + x + b) \wedge (a + y + b) \text{ for all } a, b, x, y \text{ in } G.$$

Definition 1.4 : A system $A = \{A, +, \leq\}$ is called a dually residuated lattice ordered group or DRl-group if

(i) $(A, +)$ is an abelian group

(ii) $(A, \leq)$ is a lattice

(iii) $b \leq c \Longrightarrow a + b \leq a + c$ for all a, b, c in A

(iv) Given a, b in A there exist least element $x = a - b$ in A such that $b + x \geq a$

Definition 1.5 : A non-empty set B is called Browerian Algebra if and only if

(i) $(B, \leq)$ is a lattice

(ii) B has a least element

(iii) To each a, b in $B,$ there exist a least $x = a - b$ in B such that $b \vee x \geq a.$

Definition 1.6 : Let L be a non empty set. 0 and 1 be the least and the greatest element of L and $\rightarrow$ be a binary operation. If $\rightarrow$ satisfies the following conditions for all $x, y, z \in L$

(I_1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(I_2) $x \rightarrow x = 1$,

(I_3) $1 \rightarrow x = x$,

(I_4) $0 \rightarrow x = 1$

(I_5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,

(I_6) $(((y \to z) \to z) \to x) \to x = (((y \to x) \to x) \to z) \to z$

then $(L, \to, 0, 1)$ is called implication algebra.

Theorem 1.3 : If L is implication algebra then L is lattice implication algebra with respect to the following

(i) $x \leq y$ iff $x \to y = 1$

(ii) $x' = x \to 0$

(iii) $x \vee y = (x \to y) \to y$

(iv) $x \wedge y = (x' \to y')'$

where $x, y, 1, 0 \in L$.

Theorem 1.4 : If L is a implication algebra then

(i) $0 \to x = 1$, $1 \to x = x$, $x \to 1 = 1$

(ii) $x' = x \to 0$

(iii) $x \to y \leq (y \to z) \to (x \to z)$

(iv) $x \vee y = (x \to y) \to y$

(v) $x \leq y \Rightarrow y \to z \leq x \to z$, $z \to x \leq z \to y$

(vi) $x \leq (x \to y) \to y$

for all $x, y, z \in L$.

Hence definitions for lattice implication algebra and implication algebra are equivalent.

Commutative *l*-group implication algebra

In this section two definitions for commutative l-group implication algebra are introduced and it is established that they are equivalent.

Definition 2.1 : A non-empty set G is called commutative l-group implication algebra if and only if

(i) $(G, +)$ is a commutative group

(ii) $(G, \to)$ is an implication algebra.

(iii) $x \leq y \Rightarrow a + x \leq a + y$

$(a \to x) \to b \geq (a \to y) \to b$

$a \to (x \to b) \geq a \to (y \to b)$ for all a, b, x, y in G.

Definition 2.2 : A non empty set G is called commutative l-group implication algebra iff

(i) $(G, +)$ is a commutative group

(ii) $(G, \to)$ is an implication algebra

(iii) $a + (x \vee y) = (a + x) \vee (a + y)$

$a + (x \wedge y) = (a + x) \wedge (a + y)$

$[a \to (x \vee y)] \to b = [(a \to x) \to b] \wedge [(a \to y) \to b] = a \to [(x \vee y) \to b]$

$[a \to (x \wedge y)] \to b = [(a \to x) \to b] \vee [(a \to y) \to b = a \to [(x \wedge y) \to b]$

for all x, y, a, b in G.

Theorem 2.1 : The above two definitions for commutative *l*-group implication algebra are equivalent

Proof : Assume that G is a commutative *l*-group implication algebra with respect to the first definition.

To prove that G is a commutative *l*-group implication algebra with respect to the second definition

That is assume that $x \le y \Rightarrow a + x \le a + \vee,\ (a \to x) \to b \ge (a \to y) \to b$

$$a \to (x \to b) \ge a \to (y \to b) \text{ for all } x, y, a, b \text{ in } G.$$

To prove

(i) $a + (x \vee y) = (a + x) \vee (a + y)$

(ii) $a + (x \wedge y) = (a + x) \wedge (a + y)$

(iii) $[a \to (x \vee y)] \to b = (a \to x \to b) \wedge (a \to y \to b)$

(iv) $[a \to (x \wedge y)] \to b = (a \to x \to b) \vee (a \to y \to b)$

Now let x, y, a, b in G be arbitrary.

For (i), We have

$$x \le x \vee y, y \le x \vee y \Rightarrow a + x \le a + (x \vee y)\ a + y \le a + (x \vee y) \quad \ldots(1)$$

Suppose

$$a + x \le a + u,\ a + y \le a + u \Rightarrow x \le u,\ y \le u \Rightarrow x \vee y \le u \Rightarrow a + (x \vee y) \le a + u \quad \ldots(2)$$

From (1) & (2), $a + (x \vee y)$ is a l.u.b of $a + x,\ a + y$

l.u.b of $a + x,\ a + y$ is $(a + x) \vee (a + y)$

Hence by uniqueness of l.u.b. $a + (x \vee y) = (a + x) \vee (a + y)$

For (ii), We have

$$x \wedge y \le x,\ x \wedge y \le y \Rightarrow a + (x \wedge y) \le a + x,\ a + (x \wedge y) \le a + y \quad \ldots(3)$$

Suppose

$$a + v \le a + x,\ a + v \le a + y \Rightarrow v \le x, v \le y \Rightarrow v \le x \wedge y \Rightarrow a + v \le a + (x \wedge y) \quad \ldots(4)$$

From (3) & (4), we have $a + (x \wedge y)$ is a g.l.b of $a + x, a + y$

g.l.b of $a + x,\ a + y$ is $(a + x) \wedge (a + y)$

Hence by uniqueness of g.l.b. $a + (x \wedge y) = (a + x) \wedge (a + y)$

For (iii), we have $x \le x \vee y,\ y \le x \vee y \Rightarrow (a \to x) \to b \ge [a \to (x \vee y)] \to b$

$$(a \to y) \to b \ge [a \to (x \vee y)] \to b \quad \ldots(5)$$

Suppose $(a \to x) \to b \ge (a \to u) \to b,\ (a \to y) \to b \ge (a \to u) \to b$

$$\Rightarrow a \to x \le a \to u, a \to y \le a \to u \Rightarrow x \le u,\ y \le u \Rightarrow x \vee y \le u$$

$$\Rightarrow a \to (x \vee y) \le a \to u \Rightarrow a \to (x \vee y) \to b \ge (a \to u) \to b \quad \ldots(6)$$

From (5) & (6), $a \to x \vee y \to b$ is a g.l.b of $(a \to x) \to b,\ (a \to y) \to b$

g.l.b of $(a \to x) \to b, (a \to y) \to b$ is $[(a \to x) \to b] \wedge [(a \to y) \to b]$

Hence by uniqueness of g.l.b., $[a \to (x \vee y)] \to b = ((a \to x) \to \text{b}) \wedge [(a \to y) \to b]$

$$x \le x \vee y,\quad y \le x \vee y \Rightarrow x \to b \ge (x \vee y) \to b,\ y \to b \ge (x \vee y) \to b$$

$\Rightarrow a \to (x \to b) \geq a \to [(x \vee y) \to b], \quad a \to (y \to b) \geq a \to [(x \vee y) \to b]$

Suppose $a \to (x \to b) \geq a \to (v \to b)$ $a \to (y \to b) \geq a \to (v \to b)$ ...(7)

$$x \to b \geq v \to b, \quad y \to b \geq v \to b$$

$x \leq v,\ y \leq v \Rightarrow x \vee y \leq v \Rightarrow (x \vee y) \to b \geq v \to b \Rightarrow a \to [(x \vee y) \to b] \geq a \to (y \to b)$ …(8)

From (7) & (8), $a \to [(x \vee y) \to b]$ is a g.l.b of $a \to (x \to b),\ a \to (y \to b)$,

g.l.b of $a \to (x \to b),\ a \to (y \to b)$ is $[a \to (x \to b)] \wedge [a \to (y \to b)]$

Hence by uniqueness of g.l.b., $a \to [(x \vee y) \to b] = [a \to (x \to b) \wedge [a \to (y \to b)]$

For (iv), We have $x \leq x \wedge y,\ y \leq x \wedge y \Rightarrow a \to x \geq a \to x \wedge y,\ a \to y \geq a \to x \wedge y$

$$(a \to x) \to b \leq a \to (x \wedge y)] \to b \quad (a \to y) \to b \leq a \to (x \wedge y)] \to b$$

Suppose $(a \to x) \to b \leq (a \to v) \to b,\ (a \to y) \to b \leq (a \to v) \to b$ …(9)

$$a \to x \geq a \to v,\ a \to y \geq a \to v \Rightarrow x \geq v,\ y \geq v \Rightarrow x \wedge y \geq v,$$

$$a \to x \wedge y \geq a \to v \ [a \to (x \wedge y)] \to b \leq a \to v \to b \quad \text{…(10)}$$

From (9) and (10) we have $[a \to (x \wedge y)] \to b$ is the l.u.b. of $(a \to x) \to b$, $(a \to y) \to b$

l.u.b. of $(a \to x) \to b,\ (a \to y) \to b$ is $[(a \to x) \to b] \vee [(a \to y) \to b]$

Hence by uniqueness of l.u.b.,

$$[a \to (x \wedge y)] \to b = [(a \to x) \to b] \vee [(a \to y) \to b]$$

Converse part: Assume that *G* is a commutative *l*-group implication algebra with respect to second definition

To prove that *G* is a commutative *l*-group implication algebra with respect to first definition

It is sufficient to prove $x \leq y \Rightarrow a + x \leq a + y \Rightarrow (a \to x) \to b \geq (a \to y) \to b$

$$a \to (x \to b) \geq a \to (y \to b) \text{ for all } a, b, x, y \text{ in } G$$

Let *a, b, x, y* in *G* be arbitrary and $x \leq y$

$$x \leq y \Rightarrow x \vee y = y,\ x \wedge y = x \quad \text{...(1)}$$

For (i), $(a+x) \vee (a+y) = a + x \vee y$, by definition (2)

$= a + y$ by (1)

$\Rightarrow a + x \leq a + y$

For (ii), $[(a \to x) \to b] \wedge [(a \to y) \to b] = [a \to (x \vee y)] \to b$ by definition (2)

$= (a \to y) \to b$

$\Rightarrow (a \to x) \to b \geq (a \to y) \to b$

For (iii), $[a \to (x \to b)] \vee [(a \to (y \to b)] = a \to [(x \wedge y)] \to b]$

$= a \to (x \to b)$

$\Rightarrow a \to (x \to b) \geq a \to (y \to b)$

Examples

In this section we have given some examples of commutative *l*-group implication algebra and established that class of commutative *l*-group implication algebras lies between class of Browerian algebras and class of *l*-groups.

Theorem 3.1 : Every Boolean ring with identity is a commutative ℓ-group implication algebra. Converse is not true.

Theorem 3.2 : Every commutative ℓ-group implication algebra is a l-group.

Theorem 3.3 : Any Browerian implication algebra is a commutative l-group implication algebra.

Example 3.4 : Let $L = \{0, a, b, c, d, 1\}$ be a set with Figure 1 as a partial ordering. Define a unary operation " ' " and a binary operation → denoted by juxt a position on L as follows (Tables 1 and 2 respectively)

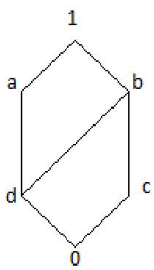

Figure 1

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

Table 1

→	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Table 2

Then implication algebra is not a commutative l-group implication algebra.

Properties of Commutative *L*-Group Implication Algebra

In this section properties of commutative l-group implication algebra are derived. The relation between commutative l-group implication algebra, Browerian Algebra and Boolean Algebra are established.

It is evident that, the commutative l-group implication algebra has the following properties:

Property 4.1 : $[(a - b) \vee 0] + b = a \vee b$ for all a, b in G.

Property 4.2 : $a \leq b \quad a - c \leq b - c$ and $c - b \leq c - a$, for all a, b, c in G.

Property 4.3 : $(a \vee b) - c = (a - c) \vee (b - c)$ for all a, b, c in G.

Property 4.4 : $a - (b \vee c) = (a - b) \wedge (a - c)$ for all a, b, c in G.

Property 4.5 : $a \geq b, \quad (a - b) + b = a$ for all a, b, c in G.

Property 4.6 : $a \vee b + a \wedge b = a + b,$ for all a, b in G.

Property 4.7 : $(a - b) \vee 0 + a \wedge b = a,$ for all a, b in G.

Property 4.8 : $a \vee b - a \wedge b = (a - b) \vee (b - a)$ for all a, b in G.

Property 4.9 : (i) $a - (b - c) \leq (a - b) + c$

$(a + b) - c \leq (a - c) + b$ for all a, b in G.

Theorem 4.1 : Any l-group implication algebra is a distributive lattice.

Theorem 4.2 : If G is a commutative l-group implication algebra and $a + b = a \vee b$ to each a, b in G then there exists least element x in G such that $b \vee x = b + x \geq a$ then G is a Browerian algebra.

Theorem 4.3 : If G is a commutative l-group implication algebra and G is a Browerian algebra then $a + b = a \vee b$ for all a, b in G.

Theorem 4.4 : If G is a commutative l-group implication algebra then

(i) $a * b \geq 0$

(ii) $a * b = 0$ iff $a = b$

(iii) $a * b = b * a$

(iv) $(a \vee b) * (a \wedge \wedge b) = a * b$ for all a, b in G.

Theorem 4.5 : If the symmetric difference is associative is a commutative l-group implication algebra G then $(G, *, \wedge)$ is a Boolean algebra and further

$$a + b = a \wedge b = a * b * (a \wedge b)$$

$$a - b = a * (a \wedge b) \text{ for all } a, b \text{ in } G.$$

CHARACTERIZATION THEOREM

In this section, to establish the characterization theorem for commutative l-group implication algebra G.

Theorem 5.1 : Any commutative l-group implication algebra G is a direct product of Browerian implication algebra and an l-group implication algebra S iff

(i) $(a + b) - (c + c) \geq (a - c) + (b - c)$ and

(ii) $(ma + nb) - (a + b) \geq (m\,a - a) + (nb - b)$

for all a, b, c in G and any positive integers m, n.

Proof : Assume that

$$(a + b) - (c + c) \geq (a - c) + (b - c) \quad \ldots (1)$$

$$(ma + nb) - (a + b) \geq (ma - a) + (nb - b)$$

for all a, b, c in G and any positive integers m, n …(2)

Then $(a + b) - (c + c) \leq a - c) + (b - c)$ and ...(3)

$$(ma + nb) - (a + b) \leq (ma - a) + (nb - b) \quad \ldots (4)$$

From (1) and (3) we have

$$(a + b) - (c + c) = (a - c) + (b - c)$$

From (2) and (4) we have

$$(ma + nb) - (a + b) = (ma - a) + (nb - b)$$

To prove $G = B \times S$

Let $B = \{a/a + a - a = 0\}$

$S = \{a/a + a - a = a\}$

Then we observe that B is a Browerian Algebra. S is a l-group implication algebra.

It is easy to prove for any a in G

$$y = (a + a) - a$$

$$x = a - [(a + a) - a]$$

$$\Rightarrow y \in S, \quad x \in B$$

and $a = x + y$ where $y \in S$, $x \in B$ in a unique way.

Hence G is the direct product of Browerian Algebra B and an l-group implication algebra S.

Conversely assume that a commutative l=group implication algebra $G = B \times S$ where B is a Browerian algbra and S is a l-group implication algebra

To prove:

$$(a + b) - (c + c) \geq (a - c) + (b - c)$$

$$(ma + nb) - (a + b) \geq (ma - a) + (nb - b)$$

for all a, b, c in G and any positive integers m, n.

Let a, b, c in G be arbitrary

$\Rightarrow$ (i) To each $[(a - c) + (b - c)]$, $a + b \in B$ there exist a least $-(c + c) \in B$

Such that $(a + b) - (c + c) \geq (a - c) + (b - c)$ and

$\Rightarrow$ (i) To each $(ma - a) + (nb - b)$, $ma + b \in B$ there exist a least $-(a + b) \in B$

Such that $(ma + nb) - (a + b) \geq (ma - a) + (nb - b)$

Since B is a Browerian algebra,

$\Rightarrow$ (i) $(a + \text{b}) - (c + c) \geq (a - c) + (b - c)$

(ii) $(ma + nb) - (a + b) \geq (ma - a) + (nb - b)$

for all a, b, c in G and any positive integers m, n.

REFERENCES

1. Ayyappan, M., and Natarajan, R., Lattice ordered near ring, *Acta Ciencia Indica*, Vol. **XXXVIII** M, No. **4**, 727 (2012).
2. Belsch, G., *Near Rings and Near Fields,* North Holland, Amsterdam (1987).
3. Birkhoff, G., Lattice Theory, *Amer. Math. Soc. Colloq., Publ*., **XXV**, Providence R.I, Third Edition, Third Printing (1979).
4. Jeyalakshmi, M. and Natarajan, R., DR *l*-group, *Acta Ciencia Indica*, Vol. **XXIX** M, No. **4**, 823-830 (2003).
5. Meenakshi, R. and Natarajan, R., *l*-module over a ring *R*, *International Journal of Algebra*, Vol. **4**, No. **18**, 851-858 (2010).
6. Natarajan, R. and Thenmozhy. R., *l*-filters in commutative *l*-group, *Acta Ciencia Indica*, Vol. **XXXIV** M, No. **3**, 1203-1216 (2008).
7. Natarajan, R. and Vimala, J., Distributive *l*-ideals in commutative *l*-group, *Acta Ciencia Indica*, Vol. **XXXIII** M, No. **2**, 517-526 (2007).

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