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ON AXIOMATIC STRUCTURE OF SET THEORY AND ITS CATEGORICAL REPRESENTATION

SHAHID AHMAD HASHMI

Department of Mathematics, Karim City College, Jamshedpur -831001

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Our main aim is to examine salient features of Wittgenstein's Categorial version of set theory which is similar to the characteristic features of lattice structure of sets. Whereas Toby Bartel's categorical version is inadequate in comparison to the classical approach of set theoretic version of sets under containment. The occurrences of "set" by "object" and of "function" by "morphism", "improper set" by "set of elements", "underlying set" by "shadow", all occurrences of "subset of" by "set in", and all other occurrences of "subset" by "set" are replaced to formulate axioms of category to show its equivalence with lattice structure of ZF axiomatic set theory.

KEYWORDS : (Morphism, Lattice, Category, Operators, Comprehension, associative and Axiom of choice).

INTRODUCTION

Let us choose a first-order dependently typed language with equality and a single ternary predicate which means that the composite of two terms equals the third, but that makes the axioms more exhaustive and full of paradoxes. We state the axioms of elementary category theory with the following notation.

(i) terms of type S (for objects);

(ii) a constant term • (the point) of type S;

(iii) for each pair of terms X and Y of type S, terms of type $F_{X,\ Y}$ (for morphisms from X to Y), and an equality predicate for terms of type $F_{X,\ Y}$, and

(iv) for each triple of terms X, Y, and Z of type S, term f of type $F_{X,\ Y}$ and term g of type FY, Z, a term $g \circ f$ (the composite of g after f) of type $F_{X,\ Z}$.

$F_{X,\ Y}$ with ": $X \rightarrow Y$". ": X" as an abbreviation of ": $\bullet \rightarrow X$"; a term of type $F\bullet_{,X}$ is an element of X.

1.2 (i) Axiom of Associativity :

$$\forall\ W, \forall\ X, \forall\ Y, \forall\ Z, \forall f: W \rightarrow X, \forall\ g: X \rightarrow Y, \forall\ h: Y \rightarrow Z, (h \circ g) \circ f = h \circ g \circ f$$

(ii) Axiom of Identities :

$$\forall\ X, \exists\ I: X \rightarrow X, \forall\ Y, (\forall f: X \rightarrow Y, f = f \circ i) \wedge \forall\ f: Y \rightarrow X, f = i \circ f$$

(iii) Axiom (Scheme) of Choice : For each formula ϕ with a variable of type $F\bullet,{}_X$, a variable of type $F\bullet,{}_Y$, and an additional variables $\vec{\chi}$ whose types depend on X and Y the axiom is given by

$$\forall X, \forall Y, \forall \vec{\chi}, (\forall a : X, \exists b : Y, \phi(a, b, \vec{\chi})) \Rightarrow \exists f : X \to Y, \forall a : X, \phi(a, f \circ a, \vec{\chi})$$

(iv) Axiom of Extensionality

$$\forall X, \forall Y, \forall f : X \to Y, \forall g : X \to Y, (\forall a : X, f \circ a = g \circ a) \Rightarrow f = g$$

(v) Axiom of the Point :

$$\forall u : \bullet, \forall v : \bullet, u = v$$

(vi) Axiom of Products :

$$\forall X, \forall Y, \exists C, \exists p : C \to X, \exists q : C \to Y, \forall a : X, \forall b : Y, \exists u : C,$$
$$a = p \circ u \wedge b = q \circ u \wedge \forall v : C, a = p \circ u \Rightarrow b = q \circ u \Rightarrow u = v$$

(vii) Axiom of Power Objects :

$$\forall X, \exists P, \exists M, \exists e : M \to X, \exists s : M \to P, \forall D, \forall I : D \to X,$$
$$\exists u : P, (\forall a : X, (\exists b : D, a = i \circ b) \Leftrightarrow \exists c : M, a = e \circ c \wedge u = s \circ c) \wedge$$
$$\forall v : P, (\forall a : X, (\exists b : D, a = i \circ b) \Leftrightarrow \exists c : M, a = e \circ c \wedge v = s \circ c) \Rightarrow u = v$$

(viii) Axiom of Infinity :

$$\exists N, \exists z : N, \exists s : N \to N, \forall a : N, \neg\ (z = s \circ a) \wedge \forall b : N, s \circ a = s \circ b \Rightarrow a = b$$

(ix) Axiom (Scheme) of Separation: For each formula ϕ with a variable of type $F\bullet,{}_X$ and additional variables $\vec{\chi}$ whose types depend on X, such that

$$\forall X, \forall \vec{\chi}, \exists S, \exists i : S \to X, (\forall a : S, \forall b : S, i \circ a = i \circ b \Rightarrow a = b) \wedge \forall a : X, \phi(a, \vec{\chi}) \Leftrightarrow \exists b : S, i \circ b = a$$

(x) Axiom (Scheme) of Collection: For each formula ϕ with a variable of type $F\bullet,{}_X$, a variable of type S, and an additional variables $\vec{\chi}$ whose types may depend on X, the axiom is expressed as follows.

$$\forall X, \forall \vec{\chi}, (\forall a : X, \exists B, \phi(a, B, \vec{\chi})) \Rightarrow \exists U, \exists p : U \to X, \forall a : X, \exists B, \phi(a, B, \vec{\chi}) \wedge \exists I : B \to U, (\forall y : B, \forall z : B, i \circ y = i \circ z \Rightarrow y = z) \wedge \forall y : U, a = p \circ y \Leftrightarrow \exists z : B, y = i \circ z$$

1.3 Theorem

A morphism $f : A \to B$ is bijective if and only if it is both injective and surjective.

Proof : Let us suppose that f is bijective. Let x and y be elements of A such that $f \circ x = f \circ y$; then the following relation hold

$$x = id_A \circ x = (f^{-1} \circ f) \circ x = f^{-1} \circ f \circ x = f^{-1} \circ f \circ y = (f^{-1} \circ f) \circ y = id_A \circ y = y, \quad \ldots(1)$$

$\Longrightarrow$ so f is injective. Let x be an element of B; then

$$f \circ f^{-1} \circ x = (f \circ f^{-1}) \circ x = id_B \circ x = x,$$

$\Longrightarrow$ so f is surjective.

Conversely, let us suppose that f is injective and surjective both By an appropriate application of the Axiom of Choice and a statement that $x = f \circ y$, we obtain the following relation

$$(\forall x : B, \exists y : A, x = f \circ y) \Rightarrow \exists g : B \to A, \forall x : B, x = f \circ g \circ x \quad \ldots(2)$$

Since f is surjective, it holds for g. If x is an element of B, then $x = f \circ g \circ x$; if y is an element of A, then $f \circ y = f \circ g \circ f \circ y$, so $y = g \circ f \circ y$ since f is injective. Therefore, g is an inverse of f.

1.4. Theorem (Morphism Comprehension Schema)

Let A and B be objects, and let $\phi\,(x, y, \vec{\chi})$ be a property of elements of A, elements of B, and some other variables. It is each element $x : A$, there exists a unique element $y : B$ such that $\phi\,(x, y, \vec{\chi})$ holds. Then there exists a unique morphism $f : A \to B$ such that, for every $x : A$ and $y : B$, $y = f \circ x$ if and only if $\phi\,(x, y, \vec{\chi})$ holds.

Proof : By an appropriate application of the Axiom of Choice to A, B, and ϕ, we get the following implication relation

$$(\forall\, x : A, \exists\, y : B, \phi\,(x, y, \vec{\chi})) \Rightarrow \exists\, f : A \to B, \forall\, x : A, \phi\,(x, f \circ x, \vec{\chi}) \qquad \ldots(3)$$

Hence, the hypothesis of theorem is satisfied to get f. we now verify that it has the property is unique. If $y = f \circ x$, then $\phi\,(x, y, \vec{\chi})$ holds; if, conversely, $\phi\,(x, y, \vec{\chi})$ holds, then $y = f \circ x$ by the condition of unicity. If g is a morphism satisfying the given property of f and x is an element of A, then $\phi\,(x, g \circ x, \vec{\chi})$ holds, so $f \circ x = g \circ x$, by using axiom of extensionality, $f = g$. Here $\psi\,(x, \vec{\chi})$ stand for an element of B, let which states that $y = \psi\,(x, \vec{\chi})$, and the morphism is denoted by $(\psi\,(x, \vec{\chi}))\; x : A$, in which x is a dummy variable. In particular, we obtain the relation

$$(\psi\,(x, \vec{\chi}))\; x : A \circ w = \psi\,(w, \vec{\chi})$$

for every element w:A; conversely, it implies that

$$(f \circ x)\; x : A = f \qquad \ldots(4)$$

1.5. Theorem

Let A, B, and G be objects. Then, for every Cartesian product $A \times B$ of A and B, and given morphisms $x : G \to A$ and $y : G \to B$, there is a unique morphism from G to $A \times B$, denoted (x, y), such that $\pi \circ (x, y) = x$ and $\rho \circ (x, y) = y$.

Proof : By an application of Morphism Comprehension to G, $A \times B$, and a statement that the values in A and B it gives rise to the expression.

$$\pi \circ u = x \circ i \wedge \rho \circ u = y \circ I \qquad \ldots(5)$$

(where i is an arbitrary element of G and u is the unique element of $A \times B$). The Axiom of Products is applied to $x \circ i$ and $y \circ i$, which gives a unique u satisfying the condition (5). We get a unique morphism $(x, y) : G \to A \times B$ such that, for every $i : G$, given by (i) – (iii) $\pi \circ (x, y) \circ i = x \circ i$ and $\rho \circ (x, y) \circ i = y \circ i$. By an application axioms of Associativity and Extensionality, $\pi \circ (x, y) = x$ and $\rho \circ (x, y) = y$. Let $e : G \to A$ be any morphism such that $\pi \circ e = x$ and $\rho \circ e = y$. Given any element $I : G$, $\pi \circ e \circ i = (\pi \circ e) \circ i = x \circ i$ similarly $\rho \circ e \circ i = y \circ i$, so $e \circ i$ satisfies the requirement for v in the Axiom of Products. Which satisfies $(x, y) \circ i$, so $(x, y) \circ i = e \circ i$; By an application of Extensionality, we get $(x, y) = e$.

Conversely, any $A \times B$, π, and ρ satisfy this universal property of a product of A and B, is obtained by taking G to be the point. In this case, we obtain the pairing $(x, y) : A \times B$ of elements $x : A$ and $y : B$.

1.6. Theorem

Let A and B be objects, and fix a Cartesian product of A and B. If C, with $p : C \to A$ and $q : C \to B$, is also a Cartesian product of A and B, then there is a unique bijection $f : C \to A \times B$ such that $p = \pi \circ f$ and $q = \rho \circ f$.

Proof : By an application of unique morphism f, denoted by (p, q); it is only required to verify that f is a bijection. Given elements x and y of C such that $f \circ x = f \circ y$, it satisfies the relation.

$$p \circ x = (\pi \circ f) \circ x = \pi \circ f \circ x = \pi \circ f \circ y = (\pi \circ f) \circ y = p \circ y \qquad \ldots(6)$$

Similarly $q \circ x = q \circ y$; since C, with p and q, is a Cartesian product, we thus obtain $x = y$; therefore, f is injective. Given an element x of $A \times B$, (since C is a Cartesian product) there exist an element $y : C$ such that $p \circ y = \pi \circ x$ and $q \circ y = \rho \circ x$. Which is the defining property of $f \circ y$, so $x = f \circ y$; therefore, f is surjective.

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