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AN ALGEBRAIC CONCEPT ON DISTRIBUTIVE SUBSEMILATTICE

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In this paper, the idea of distributive filters is introduced based on subsemilattices. Some theorems are established. Further, generalization of distributive filters for convex subsemilattices, called distributive subsemilattice is introduced.

KEYWORDS : Lattice, distributive lattice, Sublattices, Distributive Filters, Convex subsemilattice.

PRELIMINARIES

A **lattice** is a poset in which any two elements have a g.l.b and l.u.b.

A lattice L is called a **distributive lattice** if a $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in L$.

A non empty subset S of a lattice L is called **sublattice** if a, b in S implies $a \vee b$, $a \vee b$ in S.

A **meet-semilattice** is **distributive**, if for all a, b, and x:

If $a \wedge b \leq x$ then there exist a' and b' such that $a \leq a'$, $b \leq b'$ and $x = a' \wedge b'$.

A **join-semilattice** is **distributive**, if for all a, b, and x:

If $x \leq a \vee b$ then there exist a' and b' such that $a' \leq a$, $b' \leq b$ and $x = a' \vee b'$.

Any distributive meet-semilattice in which binary joins exist is a distributive lattice. A join-semilattice is distributive if and only if the lattice of its ideals is distributive.

A filter F of a lattice is called **distributive filter**

if $F \vee (X \wedge Y) = (F \vee X) \wedge (F \vee Y)$ for all X, Y in $F(L)$.

Let S be a semilattice and D a non-empty subset of S, then D is called a **convex subsemilattice** if,

(i) $\boldsymbol{a}, \boldsymbol{b} \in D \Rightarrow a \vee b \in D$, (ii) $\boldsymbol{x}, \boldsymbol{y} \in D$, $c \in s$ and $x \leq c \leq y \Rightarrow c \in D$

A convex subsemilattice is generated by a subset A of a semilattice S will be denoted $\langle A \rangle$

For any two non empty subsets of A and B of a semilattice S, it is defined that

$$A \vee B = <\{a \vee b / a \in A, b \in B\}> \text{ and } A \wedge B = <\{x / x \in A, x \in B\}>$$

That is $A \vee B$ and $A \wedge B$ are convex subsemilattices of S generated by the elements $a \vee b$ and $x = a = b$ (where $a \in A$, $b \in B\}$ respectively.

Theorem 1. For each $d \in S$, $\{d\}$ is a distributive convex subsemilattice of S.

Proof : Take $D = \{d\}$

$$D \cap X \neq \phi \Rightarrow d \in X \Rightarrow <D, X> = X \quad \ldots (1)$$

$$D \cap Y \neq \phi \Rightarrow d \in Y \Rightarrow <D, Y> = Y \quad \ldots (2)$$

Now $d \in X$, $d \in Y$ implies $d \in X \wedge Y$ and $d \in X \vee Y$

$$\Rightarrow \quad <D, X \wedge Y> = X \wedge Y$$

and

$$<D, X \vee Y> = X \vee Y$$

Using (1) & (2)

So $\quad <D, X \wedge Y> = X \wedge Y = <D, X> \wedge <D, Y>$ and

$$<D, X \vee Y> = X \vee Y = <D, X> \vee <D, Y>$$

Whenever $D \cap X \neq \phi$ and $D \cap Y \neq \phi$.

Hence $\{d\}$ is a distributive convex subsemilattice of S.

Theorem 2. A filter F of a semilattice S is distributive if and only if it is a distributive convex subsemilattice of S.

Proof : Assume that a filter F is a distributive convex subsemilattice.

To prove : F is a distributive filter

Let X, Y be any two arbitrary filters of S

Then X, Y are convex subsemilattice of S

Moreover $\quad F \cap X \supseteq \{1\} \neq \phi$

$$F \cap Y \supseteq \{1\} \neq \phi$$

Then we have by definition of convex subsemilattice.

$$<F, X \wedge Y> = <F, X> \wedge <F, Y>$$

$$<F, X \vee Y> = <F, X> \vee <F, Y>$$

Since $<X, Y> = X \vee Y$ for the filter X, Y of S, so we arrive at

$$F \vee (X \wedge Y) = (F \vee X) \wedge (F \vee Y) \text{ for all } X, Y \text{ of } F(S)$$

$\Rightarrow$ F is a distributive filter

Conversely, let a filter F be a distributive filter of a semilattice S.

To prove F is a distributive convex subsemilattice.

Let X, Y be any two arbitrary convex subsemilattice of S.

We have $\quad [X \wedge Y) = [X) \vee [Y)$

Claim : $\quad <F, X \wedge Y> = <F, X> \wedge <F, Y>$

$$F \leq <F, X>, \; F \leq <F, Y>$$

$\Rightarrow \quad F \wedge F \leq \langle F, X \rangle \wedge \langle F, Y \rangle$

$\Rightarrow \quad F \leq \langle F, X \rangle \wedge \langle F, Y \rangle$ … (1)

$X \leq \langle F, X \rangle, Y \leq \langle F, Y \rangle$

$\Rightarrow \quad X \wedge Y \leq \langle F, X \rangle \wedge \langle F, Y \rangle$ … (2)

Therefore from (1) and (2),

$\langle F, X \wedge Y \rangle \leq \langle F, X \rangle \wedge \langle F, Y \rangle$ … (3)

Clearly,

$\langle F, X \rangle = F \vee [X)$

$\langle F, Y \rangle = F \vee [Y)$

and $\langle F, X \wedge Y \rangle = F \vee [X \wedge Y)$

Clearly $\langle F, X \rangle \wedge \langle F, Y \rangle$ is a convex subsemilattice

Let $t \in \langle F, X \rangle \wedge \langle F, Y \rangle$

$\Rightarrow \quad t = a \wedge b$ where $a \in \langle F, X \rangle, \ b \in \langle F, Y \rangle$

$\Rightarrow \quad t = a \wedge b, \quad a = f_1 \wedge x_1$ with $f_1 \in F, \quad x_1 \geq x, \ x$ in X

$b = f_2 \wedge y_1$ with $f_2 \in F, \quad y_1 \geq y, \ y$ in Y

$\Rightarrow \quad t = a \wedge b, \ a \wedge b = f_1 \wedge f_2 \wedge x_1 \wedge y_1$ with $f_1 \wedge f_2 \in F$

$x_1 \wedge y_1 \geq x \wedge y, \quad x \wedge y \in X \wedge Y$

$\Rightarrow \quad t = a \wedge b, a \wedge b = (f_1 \wedge f_2) \wedge (x_1 \wedge y_1)$

with $(f_1 \wedge f_2) \wedge (x_1 \wedge y_1) \in (F, X \wedge Y)$

$t \in \langle F, X \wedge Y \rangle$

Therefore , $\langle F, X \rangle \wedge \langle F, Y \rangle \leq \langle F, X \wedge Y \rangle$ …(4)

From (3) and (4)

$\langle F, X \rangle \wedge \langle F, Y \rangle = \langle F, X \wedge Y \rangle$

Next we claim that $\langle F, X \vee Y \rangle = \langle F, X \rangle \vee \langle F, Y \rangle$

We have

$F \leq \langle F, X \rangle, \quad F \leq \langle F, Y \rangle$

$\Rightarrow \quad F \vee F \leq \langle F, X \rangle \vee \langle F, Y \rangle$

$\Rightarrow \quad F \leq \langle F, X \rangle \vee \langle F, Y \rangle$

Also $X \leq \langle F, X \rangle, \ Y \leq \langle F, Y \rangle$

$\Rightarrow \quad X \vee Y \leq \langle F, X \rangle \vee \langle F, Y \rangle$

Therefore $\langle F, X \vee Y \rangle \leq \langle F, X \rangle \vee \langle F, Y \rangle$ ... (5)

Let $t \in \langle F, X \rangle \vee \langle F, Y \rangle$ be arbitrary

$\Rightarrow \quad t \in (F \vee [X)) \vee (F \vee [Y))$

$\Rightarrow \quad t \geq a \wedge b$, with $a \in F \vee [X)$

$b \in F \vee [Y)$

$\Rightarrow \quad t \geq a \wedge b$, with $a \geq f_1 \wedge x_1,\ f_1 \in F, x_1 \geq x,\ \ x \in X$

$b \geq f_2 \wedge y_1, f_2 \in F,\ y_1 \geq y,\ y \in Y$

$\Rightarrow \quad t \geq (f_1 \wedge f_2) \wedge (x_1 \wedge y_1)$

with $f_1 \wedge f_2 \in$ F , $x_1 \wedge y_1 \geq x \wedge y,\ x \wedge y \in X \vee Y$

$\Rightarrow \quad t \in \ < F, X \vee Y >$

Therefore, $\quad < F, X > \vee < F, Y > \ \leq \ < F, X \vee Y >$ …(6)

From (5) and (6)

$$< F, X > \vee < F, Y > = < F, X \vee Y > \text{ for all } X, Y \in \mathcal{F}(S)$$

Hence F is a distributive convex subsemilattice of S.

Theorem 3. A dual filter D of a semilattice S is distributive if and only if it is distributive convex subsemilattice of S.

Proof : Assume that the dual filter D of a semilattice S is a distributive convex subsemilattice of S. To prove that D is a distributive dual filter.

Let X, Y be two arbitrary dual filters of S. Then the dual filter X, Y are Convex subsemilattices.

Moreover $\quad D \cap X \supseteq \{0\} \neq \phi$

$D \cap Y \supseteq \{0\} \neq \phi.$

$< D, X \wedge Y > = < D, X > \wedge < D, Y >$ …(1)

$< D, X \vee Y > = < D, X > \vee < D, Y >$ …(2)

Since $\quad < X, Y > = X \vee Y$ …(3)

for the dual filter X, Y of S. so we arrive

Using (3) in (1)

$D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y)$ …(4)

for all $X, Y \in \mathcal{F}(S)$

$D \vee (X \vee Y) = (D \vee X) \vee (D \vee Y)$

Equation (4) gives that D is a distributive dual filter.

Conversely, let D be a distributive dual filter of a semilattice S.

To prove that D is a distributive convex subsemilattice of S.

Using the obvious equality

$(X \wedge Y] = (X] \wedge (Y]$ valid for any subsets X, Y of S.

We have for convex subsemilattices of X, Y of S

Now $\quad < D, X \wedge Y > = D \vee (X \wedge Y]$

$= D \vee ((X] \wedge (Y])$

$= (D \vee (X]) \wedge (D \vee (Y])$

$$< D, X > \wedge < D, Y > = < D, X > \vee < D, Y >$$

And the distributive dual filter property gives

$$< D, X \wedge Y > = < D, X > \wedge < D, Y >$$

Next we claim that (2) is valid for every dual filter D of S

$$D \leq < D, X > \quad \text{and} \quad D \leq < D, Y >$$

$$\Rightarrow \qquad D \vee D \leq < D, X > \vee < D, Y >$$

$$\Rightarrow \qquad D \leq < D, X > \vee < D, Y >$$

$$X \leq < D, X > \quad \text{and} \quad Y \leq < D, Y >$$

Implies that $\quad X \vee Y \leq < D, X > \vee < D, Y >$

Therefore $\quad < D, X \vee Y > \leq < D, X > \vee < D, Y >$

Now $\quad < D, X > = D \vee (X]$

$$< D, Y > = D \vee (Y]$$

$$\Rightarrow \qquad < D, X > \vee < D, Y > = (D \vee (X]) \vee (D \vee (Y])$$

and $\quad < D, X \vee Y > = D \vee (X \vee Y]$

Clearly $< D, X > \vee < D, Y >$ is a convex subsemilattice generated by the elements of the form $(d_1 \vee x_1) \vee (d_2 \text{ v } y_1)$ where d_1, d_2 in D, x_1 in x, x in X and $y_1 \leq y$, y in Y

$$\Rightarrow \qquad x_1 \vee y_1 \text{ in } < D, X \vee Y >, \ (d_1 \vee d_2) \vee (x \vee y) \text{ in } < D, X \vee Y >$$

By the convexity of $< D, X \vee Y >$

$$x_1 \vee y_1 \leq (d_1 \vee x_1) \vee (d_2 \vee y_1) \leq (d_1 \vee x) \vee (d_2 \vee y)$$
$$= (d_1 \vee d_2) \vee (x \vee y)$$

Implies $(d_1 \vee x_1) \vee (d_2 \vee y_1)$ in $< D, X \vee Y >$

Thus $\quad < D, X \vee Y > = < D, X > \vee < D, Y >$

Hence D is a distributive convex subsemilattice.

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