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KAEHLERIAN WEYL-CONFORMAL AND WEYL-CONCIRCULAR RECURRENT AND SYMMETRIC SPACES OF SECOND ORDER

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In the present paper, I have been studied and defined Kaehlerian Weyl-Conformal and Weyl-Concircular recurrent and symmetric spaces of second order. Several theorems also have been investigated and proved therein. The necessary and sufficient condition for a ${}^2T_n$-space to be ${}^2P_n$-space and ${}^2C_n$-space to be ${}^2Z_n$-space have been established.

INTRODUCTION

Okumura (1962), studied some remarks on space with a certain contact structure. Tachibana (1967) and Mathai (1969) studied and defined the Bochner curvature tensor and Kaehlerian recurrent spaces respectively. Singh (1971), studied on Kaehlerian spaces with recurrent Bochner curvature tensor. Singh (1971-72) studied and defined Kaehlerian recurrent and Ricci-recurrent spaces of second order. Rawat and Dobhal (2009), studied on Einstein Kaehlerian s-recurrent space. Rawat and Kumar (2009), studied on curvature collineations in a Tachibana recurrent space. Further, Rawat and Prasad (2010), studied on holomorphically Projectively flat parabolically Kaehlerian spaces.

An n (= $2m$) dimensional Kaehlerian space K_n is an even dimensional Riemannian space, which admits a structural tensor field F_i^h satisfying the conditions ([3])

$$F_i^h \; F_h^j \; = \; -\,\delta_i^j \qquad \dots (1.1)$$

$$F_{ij} \; = \; -F_{ji} \,, \; (F_{ij} \; = \; F_i^a \; g_{aj}) \qquad \dots (1.2)$$

$$\nabla_j \; F_i^h = 0. \qquad \dots (1.3)$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemannian curvature tensor, which we denote by R_{ijk}^h is given by

$$R_{ijk}^h \; = \; \partial_i \begin{Bmatrix} h \\ j \;\; k \end{Bmatrix} - \; \partial_j \begin{Bmatrix} h \\ i \;\; k \end{Bmatrix} + \begin{Bmatrix} h \\ i \;\; l \end{Bmatrix}\begin{Bmatrix} l \\ j \;\; k \end{Bmatrix} - \begin{Bmatrix} h \\ j \;\; l \end{Bmatrix}\begin{Bmatrix} l \\ i \;\; k \end{Bmatrix} \qquad \dots (1.4)$$

where $\partial_j \; = \; \dfrac{\partial}{\partial x^i}$ and $\{x^i\}$ denotes real local coordinates.

The Ricci-tensor and scalar curvature are respectively given by

$$R_{ij} = \; R_{aij}^a \quad \text{and} \quad R = \; R_{ij} g^{ij}$$

It is well known that these tensors satisfies the following identities

$$R^a_{ijk,a} = R_{jk,i} - R_{ik,j} , \quad \text{... (1.5)}$$

$$R_{,i} = 2R^a_{i,a} \quad \text{... (1.6)}$$

$$F^a_i R_{aj} = -R_{ia} F^a_j \quad \text{... (1.7)}$$

and

$$F^a_i R^j_a = R^a_i F^j_a . \quad \text{... (1.8)}$$

The Holomorphically projective curvature tensor (Sinha 1973), Tachibana H-Concircular curvature tensor, Weyl-Conformal curvature tensor and Weyl-Concircular curvature (Sinha 1971) are respectively given by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{(n+2)} (R_{ik}\, \delta^h_j - R_{jk}\, \delta^h_i + S_{ik}\, F^h_j - S_{jk}\, F^h_i + 2S_{ij} F^h_k) , \quad \text{...(1.9)}$$

$$T^h_{ijk} = R^h_{ijk} + \frac{1}{(n+2)} (g_{ik}\, \delta^h_j - g_{jk}\, \delta^h_i + F_{ik}\, F^h_j - F_{jk}\, F^h_i + 2F_{ij} F^h_k), \quad \text{... (1.10)}$$

$$C^h_{ijk} = R^h_{ijk} + \frac{1}{(n+2)} (R_{ik}\, \delta^h_j - R_{jk} \delta^h_i + g_{ik}\, R^h_j - g_{jk}\, R^h_i) - \frac{R}{(n-1)(n-2)} (g_{ik} \delta^h_j - g_{jk}\, \delta^h_i) \quad \text{... (1.11)}$$

and

$$Z^h_{ijk} = R^h_{ijk} + \frac{R}{n(n-1)} (g_{ik}\, \delta^h_j - g_{jk}\, \delta^h_i) \quad \text{...(1.12)}$$

where

$$S_{ij} = F^a_i R_{aj} .$$

KAEHLERIAN RECURRENT SPACES OF SECOND ORDER

Definition (2.1): A Kaehler space K_n is said to be a Kaehlerian recurrent space of second order, if the following condition is satisfied (Singh 1971).

$$\nabla_m \nabla_n R^h_{ijk} = \lambda_{mn} R^h_{ijk} , \quad \text{... (2.1)}$$

where λ_{mn} is non-zero and in general, non-symmetric covariant tensor of order 2. It will be denoted briefly by ${}^2K_n - space$.

The space is said to be Ricci-recurrent space of second order, if it satisfies the condition

$$\nabla_m \nabla_n R_{ij} = \lambda_{mn} R_{ij}, \quad \text{...(2.2)}$$

Multiplying the above equation by g^{ij}, we get

$$\nabla_m \nabla_n R = \lambda_{mn} R \quad \text{...(2.3)}$$

Remark (1.1) : From (2.2), it follows that every Kaehlerian recurrent space of second order is Kaehlerian Ricci-recurrent space of second order, but converse is not necessarily true.

Definition (2.2) : A Kaehler space satisfying the condition

$$\nabla_m \nabla_n P^h_{ijk} = \lambda_{mn} P^h_{ijk} , \quad \text{... (2.4)}$$

for some non- zero tensor λ_{mn}, will be called Kaehlerian Projective recurrent space of second order, or briefly ${}^2P_n - space$.

Definition (2.3) : A Kaehler space satisfying the condition

$$\nabla_m \nabla_n T^h_{ijk} = \lambda_{mn} T^h_{ijk} , \qquad ...(2.5)$$

for some non-zero tensor λ_{mn}, will be called Kaehlerian Tachibana *H*-Concircular recurrent space of second order, or briefly $^2T_n - space$.

Definition (2.4) : A Kaehler space satisfying the condition

$$\nabla_m \nabla_n C^h_{ijk} = \lambda_{mn} C^h_{ijk} , \qquad ...(2.6)$$

for some non- zero tensor λ_{mn} , will be called Kaehlerian Weyl-Conformal recurrent space of second order, or briefly $^2C_n - space$.

Definition (2.5) : A Kaehler space satisfying the condition

$$\nabla_m \nabla_n Z^h_{ijk} = \lambda_{mn} Z^h_{ijk} , \qquad ...(2.7)$$

for some non- zero tensor λ_{mn} , will be called Kaehlerian Weyl-Concircular recurrent space of second order, or briefly $^2Z_n - space$.

Now, If we put

$$L_{ij} = R_{ij} - \frac{R}{n} g_{ij} \qquad ... (2.8)$$

and

$$M_{ij} = F^a_i L_{aj} = S_{ij} - \frac{R}{n} F_{ij} \qquad ...(2.9)$$

then from (1.9), (1.10), (2.8) and (2.9), we get

$$P^h_{ijk} = T^h_{ijk} + \frac{1}{(n+2)} (L_{ik} \delta^h_j - L_{jk} \delta^h_i + M_{ik} F^h_j - M_{jk} F^h_i + 2M_{ij} F^h_k), \quad ...(2.10)$$

and with the help of (1.11), (1.12), (2.8) and (2.9), we get

$$C^h_{ijk} = Z^h_{ijk} + \frac{1}{(n+2)} (L_{ik} \delta^h_j - L_{jk} \delta^h_i + g_{ik} L^h_j - g_{jk} L^h_i) , \qquad ...(2.11)$$

Now, we have the following:

Theorem (2.1): If a Kaehler space satisfies any two of the following properties.

(i) the space is Kaehlerian Ricci recurrent space of second order,

(ii) the space is a Kaehlerian Projective recurrent space of second order,

(iii) the space is Kaehlerian Tachibana H-Concircular recurrent space of second order , then it must also satisfy the third.

Proof: Differentiating (2.10) covariantly with respect to x^n , we have

$$\nabla_n P^h_{ijk} = \nabla_n T^h_{ijk} + \frac{1}{(n+2)} (\nabla_n L_{ik} \delta^h_j - \nabla_n L_{jk} \delta^h_i + \nabla_n M_{ik} F^h_j - \nabla_n M_{jk} F^h_i + 2\nabla_n M_{ij} F^h_k) \qquad ... (2.12)$$

Again , differentiating (2.12) covariantly with respect to x^m , we get

$$\nabla_m \nabla_n P^h_{ijk} = \nabla_m \nabla_n T^h_{ijk} + \frac{1}{(n+2)} (\nabla_m \nabla_n L_{ik} \delta^h_j - \nabla_m \nabla_n L_{jk} \delta^h_i + \nabla_m \nabla_n M_{ik} F^h_j$$

$$-\nabla_m \nabla_n M_{jk} F^h_i + 2\nabla_m \nabla_n M_{ij} F^h_k) \qquad ... (2.13)$$

Transvecting (2.10) with λ_{mn} and subtracting from (2.13), we get

$$\nabla_m\nabla_n P^h_{ijk} - \lambda_{mn} P^h_{ijk} = \nabla_m\nabla_n T^h_{ijk} - \lambda_{mn} T^h_{ijk} + \frac{1}{(n+2)}[(\nabla_m\nabla_n L_{ik} - \lambda_{mn} L_{ik})\delta^h_j$$

$$-(\nabla_m\nabla_n L_{jk} - \lambda_{mn} L_{jk})\delta^h_i + (\nabla_m\nabla_n M_{ik} - \lambda_{mn} M_{ik})F^h_j$$

$$-(\nabla_m\nabla_n M_{jk} - \lambda_{mn} M_{jk})F^h_i + 2(\nabla_m\nabla_n M_{ij} - \lambda_{mn} M_{ij})F^h_k] \qquad \text{... (2.14)}$$

The statement of the above theorem follows in view of equations (2.2), (2.3), (2.4), (2.5), (2.8), (2.9), and (2.14).

Theorem (2.2): If a Kaehler space satisfies any two of the following properties.

(i) the space is Kaehlerian Ricci recurrent space of second order,

(ii) the space is a Kaehlerian Weyl-conformal recurrent space of second order,

(iii) the space is Kaehlerian Weyl-Concircular recurrent space of second order then it must also satisfy the third.

Proof : Kaehlerian Ricci recurrent space of second order, Kaehlerian Weyl-conformal recurrent space of second order and Kaehlerian Weyl-Concircular recurrent space of second order are respectively characterized by the equations (2.2), (2.6) and (2.7).

Differentiating (2.11) covariantly with respect to x^n , we have

$$\nabla_n C^h_{ijk} = \nabla_n Z^h_{ijk} + \frac{1}{(n+2)}(\nabla_n L_{ik}\,\delta^h_j - \nabla_n L_{jk}\,\delta^h_i + \nabla_n g_{ik}\,L^h_j - \nabla_n g_{jk}\,L^h_i), \qquad \text{... (2.15)}$$

Again, differentiating (2.15) covariantly with respect to x^m , we get

$$\nabla_m\nabla_n C^h_{ijk} = \nabla_m\nabla_n Z^h_{ijk} + \frac{1}{(n+2)}(\nabla_m\nabla_n L_{ik}\,\delta^h_j - \nabla_m\nabla_n L_{jk}\,\delta^h_i + \nabla_m\nabla_n g_{ik}\,L^h_j$$

$$-\nabla_m\nabla_n g_{jk}\,L^h_i) \qquad \text{...(2.16)}$$

Transvecting (2.11) with λ_{mn} and subtracting from (2.16), we get

$$\nabla_m\nabla_n C^h_{ijk} - \lambda_{mn} C^h_{ijk} = \nabla_m\nabla_n Z^h_{ijk} - \lambda_{mn} Z^h_{ijk} + \frac{1}{(n+2)}[(\nabla_m\nabla_n L_{ik} - \lambda_{mn} L_{ik})\delta^h_j$$

$$-(\nabla_m\nabla_n L_{jk} - \lambda_{mn} L_{jk})\,\delta^h_i + (\nabla_m\nabla_n L^h_j - \lambda_{mn} L^h_j)g_{ik}$$

$$-(\nabla_m\nabla_n L^h_i - \lambda_{mn} L^h_i)g_{jk}], \qquad \text{...(2.17)}$$

The statement of the above theorem follows in view of equations (2.2), (2.3), (2.7), (2.8), (2.9), and (2.17).

Theorem (2.3): The necessary and sufficient condition that a ${}^2T_n - space$ to be ${}^2P_n - space$ is that

$$(\nabla_m\nabla_n L_{ik} - \lambda_{mn} L_{ik})\delta^h_j - (\nabla_m\nabla_n L_{jk} - \lambda_{mn} L_{jk})\delta^h_i + (\nabla_m\nabla_n M_{ik} - \lambda_{mn} M_{ik})F^h_j$$

$$-(\nabla_m\nabla_n M_{jk} - \lambda_{mn} M_{jk})F^h_i + 2(\nabla_m\nabla_n M_{ij} - \lambda_{mn} M_{ij})F^h_k = 0,$$

Proof: The statement of the above theorem follows in view of equations (2.2), (2.3), (2.4), (2.5), (2.14), similarly , we have the following

Theorem (2.4): The necessary and sufficient condition that a ${}^2Z_n - space$ to be ${}^2C_n - space$ is that

$$(\nabla_m \nabla_n L_{ik} - \lambda_{mn} L_{ik})\delta_j^h - (\nabla_m \nabla_n L_{jk} - \lambda_{mn} L_{jk})\delta_i^h + (\nabla_m \nabla_n L_j^h - \lambda_{mn} L_j^h)g_{ik} - (\nabla_m \nabla_n L_i^h - \lambda_{mn} L_i^h)g_{jk}] = 0.$$

Proof: The statement of the above theorem follows in view of equations (2.2), (2.3), (2.6), (2.7), and (2.17).

KAEHLERIAN SYMMETRIC SPACES OF SECOND ORDER

Definition (3.1) : A Kaehlerian space is said to be Kaehlerian symmetric space of second order, if it satisfies

$$\nabla_m \nabla_n R_{ijk}^h = 0, \quad \text{or} \quad \text{equivalently} \quad \nabla_m \nabla_n R_{ijkl} = 0, \qquad ...(3.1)$$

and it is called Ricci-symmetric space of second order of

$$\nabla_m \nabla_n R_{ij} = 0, \qquad ...(3.2)$$

Multiplying the above by g^{ij}, we get

$$\nabla_m \nabla_n R = 0 \qquad ...(3.3)$$

Definition (3.2) : A Kaehler space satisfying the condition

$$\nabla_m \nabla_n P_{ijk}^h = 0, \text{ or , equivalently } \nabla_m \nabla_n P_{ijkl} = 0, \qquad ...(3.4)$$

is called a Kaehlerian Projective symmetric space of second order.

Definition (3.3) : A Kaehler space satisfying the condition

$$\nabla_m \nabla_n T_{ijk}^h = 0, \quad \text{or} \quad \text{equivalently} \quad \nabla_m \nabla_n T_{ijkl} = 0, \qquad ...(3.5)$$

is called a Kaehlerian Tachibana H-Concircular symmetric space of second order.

Definition (3.4) : A Kaehler space satisfying the condition

$$\nabla_m \nabla_n C_{ijk}^h = 0, \text{ or , equivalently } \nabla_m \nabla_n C_{ijkl} = 0, \qquad ...(3.6)$$

is called a Kaehlerian Weyl-Conformal symmetric space of second order.

Definition (3.5) : A Kaehler space satisfying the condition

$$\nabla_m \nabla_n Z_{ijk}^h = 0, \text{ or equivalently } \nabla_m \nabla_n Z_{ijkl} = 0, \qquad ...(3.7)$$

is called a Kaehlerian Weyl-Concircular symmetric space of second order.

Theorem (3.1): If a Kaehler space satisfies any two of the following properties:

(i) the space is Kaehlerian Ricci -symmetric space of second order,

(ii) the space is a Kaehlerian Projective symmetric space of second order,

(iii) the space is Kaehlerian Tachibana H-Concircular symmetric space of second order then it must also satisfy the third.

Proof: In a Kaehlerian Ricci -symmetric space of second order, the condition (3.2) is satisfied, whereas, Kaehlerian Projective symmetric space of second order and a Kaehlerian Tachibana H-Concircular symmetric space of second order respectively are given by the conditions (3.4) and (3.5). Therefore the statement of the above theorem follows in the view of equations (2.13), (3.2), (3.4) and (3.5).

Theorem (3.2) : If a Kaehler space satisfies any two of the following properties:

(i) the space is Kaehlerian Ricci -symmetric space of second order,

(ii) the space is a Kaehlerian Weyl-conformal symmetric space of second order,

(iii) the space is Kaehlerian Weyl-Concircular symmetric space of second order then it must also satisfy the third.

Proof : In a Kaehlerian Ricci -symmetric space of second order, the condition (3.2) is satisfied, whereas, Kaehlerian Weyl-conformal symmetric space of second order and a Kaehlerian Weyl-Concircular symmetric space of second order respectively are given by the conditions (3.6) and (3.7).

Therefore, the statement of the above theorem follows in the view of equations (2.16), (3.2), (3.6) and (3.7).

Theorem (3.3) : The necessary and sufficient condition that a Kaehlerian Tachibana *H*-Concircular symmetric space of second order to be a Kaehlerian Projective symmetric space of second order is that

$$\nabla_m \nabla_n L_{ik} \delta_j^h - \nabla_m \nabla_n L_{jk} \delta_i^h + \nabla_m \nabla_n M_{ik} F_j^h - \nabla_m \nabla_n M_{jk} F_i^h + 2\nabla_m \nabla_n M_{ij} F_k^h = 0 ,$$

Theorem (3.4): The necessary and sufficient condition that a Kaehlerian Weyl-Concircular symmetric space of second order to be a Kaehlerian Weyl symmetric space of second order is that

$$\nabla_m \nabla_n L_{ik} \delta_j^h - \nabla_m \nabla_n L_{jk} \delta_i^h + \nabla_m \nabla_n g_{ik} L_j^h - \nabla_m \nabla_n g_{jk} L_i^h = 0.$$

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