Acta Ciencia Indica, Vol. XLI P, No. 2, 73 (2015)

STUDY OF VORTICITY OF LAMINAR FLOW WITH VARIABLE PERMEABILITY

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RECEIVED : 3 July, 2015

Introduction

The porous medium is in fact a non-homogenous medium but for the sake of analysis, it may be possible to replace it with a homogenous fluid, which has dynamical properties equal to those of non-homogenous continuum. In the continuum approach to transportation processes in porous media, the differential equation governing the macroscopic fluid motion is based on the experimentally established Darcy's empirical law (1856), which expresses the fact that the pressure gradient pushes the fluid in the porous medium against the body force on it exerted by the fluid.

The flow of the fluid is an open enclined channel with a free surface under gravity has a wide application in the design of drainage, flood discharge channels, irrigation canals, coating to the paper rolls and many others. This kind of flow has long been studied experimentally by a French engineer, Antoine Chezy whose work was brought to light by Girard (1803), Manning (1890) and Herschal (1897) long after his death. Several interesting empirical results have been reported by many researches viz. Franzini and Chisholm (1863), Darcy and Bazin (1865). Vanoni (1941), Powell (1946, 50) etc. Satya prakash (1971) has made good attempt in his work on analytical treatment of steady laminar viscous flow of a liquid down an open inclined channel under gravity with the free surface exposed to atmospheric pressure and with impermeable bed. Around same time some investigators studied problems involving permeable boundaries in the fileds of Agricultural Engineering and Petroleum industry. Ahmadi and Manavi (1971) have derived general equation of motion and applied the results obtained to some basic flow problems. Gulab Ram and Mishra (1977) applied the equation to study the MHD flow of conducting fluid through porous media. Verma and Vyas (1980) have studied viscous flow on open inclined channel with naturally permeable bed with uniform porosity. Sinha and Chadda (1980) investigated the viscous flow down an open inclined channel with a bed of varying permeability. Mittal *et al*. (1977, 1998) studied the vorticity of slow steady MHD flow of viscous conducting fluid down an inclined porous conducting plane.

The study of vorticity of steady viscous incompressible fluid through a porous medium bounded by an open inclined channel with impermeable walls has been made when it free surface is exposed to atmospheric pressure under the action of gravity. The analytical expression for vorticity has been obtained from velocity of fluid flow. To visualize the Physical behavior of the problem, numerical solution for effect of permeability and pressure gradient on vorticity have also been shown graphically through vorticity profile for different values of pressure gradient and permeability parameter, *K*.

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Mathematical formulation of the problem

Let us consider the flow of viscous in-compressible fluid of density, ρ and dynamic viscosity, μ down an open inclined channel of width $2a$ and depth h, having its side walls normal to the plane of the bed. The plane of bed is taken to be inclined at an angle, $\beta(0<\beta<\pi/2)$ with the horizontal and is of variable permeability. Let X-axis be along the central line in the direction of flow at the free surface, Y-axis along the depth and Z-axis along the width of channel. The flow of fluid under consideration is due to a constant pressure gradient at the mouth of channel.

The equations of motion describing the flow of viscous, incompressible fluid down open inclined channels are:

$$0=-\frac{\partial p}{\partial x}+\rho g\sin\beta+\mu\left(\frac{\partial^2 u}{\partial^2 y}+\frac{\partial^2 u}{\partial^2 z}\right) \qquad \ldots (1)$$

$$0=\frac{\partial p}{\partial y}+\rho g\cos\beta \qquad \ldots (2)$$

$$0=-\frac{\partial p}{\partial z} \qquad \ldots (3)$$

$$\frac{\partial u}{\partial x}=0 \qquad \ldots (4)$$

The flow in the porous media is governed by the Darcy's empirical law (1856),

$$Q=\frac{K_0}{\mu}\left(-\frac{\partial p}{\partial x}+\rho g\sin\beta\right) \qquad \ldots (5)$$

where : Q : Velocity in porous medium

K_0 : Permeability of the porous medium

p : Static pressure

g : Acceleration due to gravity

Here we have assume variable permeability, K_0 follows the exponential law

$$K_0=Ke^{-\sigma y} \qquad \ldots (6)$$

Subjected to following boundary conditions:

$$u=0 \text{ at } z=\pm a. \qquad \ldots (7)$$

$$\frac{\partial u}{\partial y}=0 \text{ at the free surface } y=0, \qquad \ldots (8)$$

Following the Beavers and Joseph (1967), the condition at the interface of the free flow region and porous medium is given by

$$\left.\frac{\partial u}{\partial y}\right|_{y=h}=\frac{a}{\sqrt{K_0}}(u_1-Q) \qquad \ldots (9)$$

where u_1 : velocity at $y = h$.

α : Dimensionless characteristic of the structure of the porous material called slip parameter.

Taking the depth, h of the channel as the characteristic length and the mean flow velocity, U to be the characteristic velocity. Let us introduce the following non-dimensional quantities.

$$x' = \frac{x}{h}, y' = \frac{y}{h}, z' = \frac{z}{h}, u' = \frac{u}{U}, K' = \frac{K}{h^2}, K_0' = \frac{K_0}{h^2}, Q' = \frac{Q}{U'} p' = \frac{p}{\rho U^2}, u_1' = \frac{u_1}{U} \text{ and } \sigma' = \sigma h \quad \ldots (10)$$

By virtue of these non-dimensional quantities equation (1), (5) and (6) reduces to (dropping the deshes for simplicity)

$$\frac{\partial^2 u}{\partial^2 y} + \frac{\partial^2 u}{\partial^2 z} = R\frac{\partial p}{\partial x} - \frac{R}{F}\sin\beta \quad \ldots (11)$$

$$Q = K_0 R\left(-\frac{\partial p}{\partial x} + \frac{1}{F}\sin\beta\right) \quad \ldots (12)$$

$$K_0 = Ke^{-\sigma y} \quad \ldots (13)$$

where $R\left(=\frac{Uh}{v}\right)$: Reynolds number, $v\left(=\frac{\mu}{\rho}\right)$: Kinematic viscosity, $F\left(=\frac{U^2}{gh}\right)$: Froude number

And changed boundary conditions are:

$$u = 0 \text{ at } z = \pm\frac{a}{h} = \pm l \text{ (say)}, \quad \ldots (14)$$

$$\frac{\partial u}{\partial y} = 0 \text{ at } y = 0, \quad \ldots (15)$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=1} = \frac{\alpha}{\sqrt{K_0}}\left(u_1 - K_0 R\frac{\partial p}{\partial x} + \frac{K_0 R}{F}\sin\beta\right), \quad \ldots (16)$$

with the help of equation (13), equation (16) becomes

$$\left.\frac{\partial u}{\partial y}\right|_{y=1} = \frac{\alpha}{\sqrt{K}.e^{-\sigma y/2}}\left[u_1 - Ke^{-\sigma y/2}R\left(\frac{\partial p}{\partial x} - \frac{1}{F}\sin\beta\right)\right] \quad \ldots (17)$$

where $u = u_1$ at $y = 1$

Solution of the problem

On putting $z = \frac{2l\xi}{\pi} - l$ in equation (11), we get

$$\frac{\partial^2 u}{\partial y^2} + \frac{\pi^2}{4l^2}\frac{\partial^2 u}{\partial z^2} = R\frac{\partial p}{\partial x} - \frac{R}{F}\sin g\beta \quad \ldots (18)$$

with changed boundary conditions

$$u = 0 \text{ at } \xi = 0 \text{ and } \xi = \pi \quad \ldots (19)$$

$$u = u_1 \text{ at } y = 1 \quad \ldots (20)$$

$$\frac{\partial u}{\partial y} = 0 \text{ at } y = 0 \quad \ldots (21)$$

Now applying finite Fourier sine transform to equation (18) and boundary conditions (20) and (21), we get

$$\frac{d^2\bar{u}}{dy^2} - \frac{n^2\pi^2}{4l^2}\bar{u} = \left(R\frac{\partial p}{\partial x} - \frac{R}{F}\sin\beta\right)\left(\frac{1-\cos n\pi}{n}\right) \quad \ldots (22)$$

$$\bar{u} = \bar{u}_1 \text{ at } y = 1 \quad \ldots (23)$$

$$\frac{d\bar{u}}{dy} = 0 \text{ at } y = 0 \quad \ldots (24)$$

where $\bar{u}(y,n) = \int_0^\pi u.\sin n\xi d\xi$, n : is positive integer

The solution of equation (22) subjected to the boundary condition (23) and (24) yields:

$$\bar{u}(y,n) = \bar{u}_1 \frac{\cosh\frac{n\pi}{2l}y}{\cosh\frac{n\pi}{2l}} + \frac{4l^2}{\pi^2 n^3}\left(-R\frac{\partial p}{\partial x} + \frac{R}{F}\sin\beta\right)(1-\cos n\pi)\left[1 - \frac{\cosh\frac{n\pi}{2l}y}{\cosh\frac{n\pi}{2l}}\right] \quad \ldots (25)$$

Applying the inversion formula for finite Fourier sine transform to the equation (25), we get

$$u(y,\xi) = \frac{2}{\pi}\sum_{n=1}^{\infty}\bar{u}(y,n).\sin n\xi$$

This gives the velocity of fluid flow

$$u = \frac{2}{\pi}\sum_{n=1}^{\infty} u_1 \frac{\cosh\frac{n\pi}{2l}}{\cosh\frac{n\pi}{2l}}.\text{sinn}\xi + \frac{8l^2}{\pi^3}\left(-R\frac{\partial p}{\partial x} + \frac{R}{F}\sin\beta\right) \times \sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n^3}\left[1 - \frac{\cosh\frac{n\pi}{2l}y}{\cosh\frac{n\pi}{2l}}\right].\text{sinn}\xi$$

$$\ldots (26)$$

On taking Finite sine transform on equation (17), we get

$$\left.\frac{\partial\bar{u}}{\partial y}\right|_{y=1} = \frac{\alpha}{\sqrt{K}}\left[\bar{u}_1.e^{\sigma/2} - Ke^{-\sigma/2}\left(R\frac{\partial p}{\partial x} - \frac{R}{F}\sin\beta\right)\left(\frac{1-\cos n\pi}{n}\right)\right] \quad \ldots (27)$$

Finding the value of $\bar{u}_1$ from equation (25) and (27) and substituting in (26), we get

$$u = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{A_n}{B_n}.\frac{\cosh\frac{n\pi}{2l}}{\cosh\frac{n\pi}{2l}}.\sin n\xi + \frac{8l^2}{\pi^3}\left(-R\frac{\partial p}{\partial x} + \frac{R}{F}\sin\beta\right)\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n^3}\left[1 - \frac{\cosh\frac{n\pi}{2l}y}{\cosh\frac{n\pi}{2l}}\right]\sin n\xi$$

$$\dots (28)$$

where $$A_n = \left(-R\frac{\partial p}{\partial x} + \frac{R}{F}\sin\beta\right)\left(\frac{1-\cos n\pi}{n}\right).\left[\frac{2l}{n\pi}\tanh\frac{n\pi}{2l} + e^{-\sigma/2}\alpha\sqrt{K}\right]$$

$$B_n = \left[\frac{n\pi}{2l}\tanh\frac{n\pi}{2l} + \frac{\alpha}{\sqrt{K}}e^{-\sigma/2}\right]$$

The vorticity of fluid flow is given by

$$\zeta = \sum_{n=1}^{\infty}\frac{A_n}{B_n}.\left(\frac{n}{l}\right).\frac{\cosh\frac{n\pi}{2l}\pi}{\cosh\frac{n\pi}{2l}}.\sin n\xi + \frac{4l}{\pi^2}\left(-R\frac{\partial p}{\partial x} + \frac{R}{F}\sin\beta\right)$$

$$\times\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n^2}\left[-\frac{\sinh\frac{n\pi}{2l}y}{\cosh\frac{n\pi}{2l}}\right].\sin n\xi \qquad \dots (29)$$

PARTICULAR CASE

When $K \to 0, l \to \infty$ and $-\frac{\partial p}{\partial x} = 0$ *i.e.* fluid is assumed to flow down an infinite inclined plane with impermeable bed, the equation (28) is in agreement with *Y*th (1963),

$$u = \frac{R}{F}\sin\beta\left[\frac{1-y^2}{2}\right] \qquad \dots (30)$$

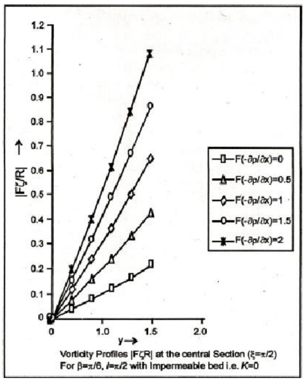
The vorticity of fluid flow in this case is given by

$$\zeta = -\frac{Ry}{F}\sin\beta \qquad \dots (31)$$

Results and discussion

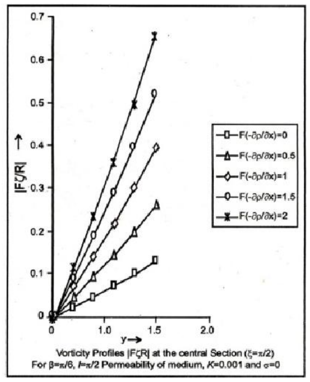
The graphs given below demonstrate the behaviour of the rotation of fluid in the flow field for different values of pressure gradients under different permeability conditions associated with the bed of channel. The Fig. 1, 2, 3 represents the vorticity profiles $\left|\frac{F\zeta}{R}\right|$ at the central section $(\xi = \pi/2)$ of the channel flow for $\beta = \pi/6$, $l = \pi/2$ due to different choices of the values of pressure gradient $F = (\partial p/\partial x) = 0$, 0.5, 1.0, 1.5, 2.0 with $K = 0.0001$, $\sigma = 0, 0.1$ and $\alpha = 0.01$.

A comparative study of these vorticity profiles (Fig. 1, 2, 3) reveals that for different values of the pressure gradient the non-zero permeability tends to decrease the vorticity of fluid flow for all points of the cross-section under consideration *i.e.* the rotation of fluid shows down all points from free surface to permeable bed than those when the bed impermeable. Further it is noticed that in case of exponentially varying permeability (Fig. 3) this tendency is comparatively induced which is due to the seepage of fluid along the permeable bed. The graphs also depicts that the increasing values of the pressure gradients increases the vorticity of fluid in the channel. It is also observed that the fluid is completely irrotational (the vorticity of fluid is zero) at the free surface of the flow and the vorticity of fluid increases as we move inside the bed and it becomes maximum at the bottom of the bed.



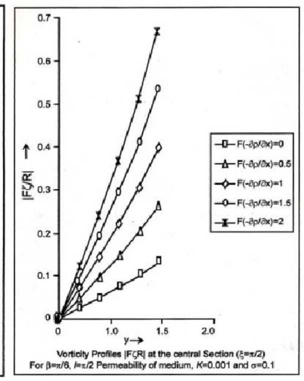
Vorticity Profiles $|F\zeta R|$ at the central Section ($\xi=\pi/2$) For $\beta=\pi/6$, $l=\pi/2$ with Impermeable bed i.e. $K=0$

Fig. 1



Vorticity Profiles $|F\zeta R|$ at the central Section ($\xi=\pi/2$) For $\beta=\pi/6$, $l=\pi/2$ Permeability of medium, K=0.001 and $\sigma=0$

Fig. 2



Vorticity Profiles $|F\zeta R|$ at the central Section ($\xi=\pi/2$) For $\beta=\pi/6$, $l=\pi/2$ Permeability of medium, K=0.001 and $\sigma=0.1$

Fig. 3

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