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GENERALIZED TOPOLOGIES RELATED TO NOTIONS OF N-CLOSED SETS AND N-CONTINUOUS FUNCTIONS

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The notion of *N*-continuous functions between generalized topological spaces has been defined. This paper relates this concept to the co *N*-closed generalized topology of a generalized topological space, and studies the properties of co *N*-closed generalized topologies. Also we consider the class of co *N*-continuous functions between generalized topological spaces.

Keywords: Generalized topology, μ-open, μ-Hausdorff, μ-regular, μ-normal, μ-continuous.

Introduction

Let X be a set and γ, a map from the power set exp X into itself. We suppose that γ is monotonic, *i.e.*, $A \subset B \subset X$ implies $\gamma A \subset \gamma B$ (which we write γA for $\gamma(A)$). We denote $\Gamma(X)$ the collection of all monotonic maps γ : exp $X \to$ exp X. In [2], a set $A \subset X$ is γ-open if and only if $A \subset \gamma A$ and it is shown in ([2], 1.1) that any union of γ-open set is γ-open. Obviously the empty set is γ-open and so γ-open set form a generalized topology.

So, if we agree in saying that a collection μ of subsets of X is a generalized topology (briefly G.T) on X if and only if $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$, implies $G = \bigcup_i G_i \in \mu$, we can say that γ-open sets constitute a G.T. If o is a topology on X in the usual sense and we denote iA the o-interior int A, by cA the o-closure cl A, we obtain an important particular cases the collection o of all open sets ($\gamma = i$), $s.o$ of all semi open sets [10] ($\gamma = ci$), $p.o$ of all pre open sets [11] ($\gamma = ic$), $\beta\, o$ of all β-open sets [1] ($\gamma = cic$), αo of all α-open sets [12] ($\gamma = ici$; the latters constitute in fact a topology finer than o).

The purpose of our paper is to formulate some simple properties of generalized topologies and to study, based on this concept, suitable generalizations of the concept of generalized continuous maps.

It is easy to show that the method of considering γ-open sets for some $\gamma \in \Gamma(X)$ can produce all G.T's on X ([2], 2.15).

Definition 1.1 [4]. Let X be a set. A subset μ of exp X is called a generalized topology on X and (X, μ) is called a generalized topological spaces [3] (abbr. GTS) if μ has the following properties:

(i) $\varphi \in \mu$,

(ii) Any union of elements of μ belongs to μ.

A generalized topology μ is said to be strong [4] (abbr. SGT) if $X \in \mu$.The elements of μ are called μ-open sets and the complement of μ-open sets are called μ-closed sets. For $A \subset X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, that is the smallest μ-closed set containing A, and by $i_\mu(A)$, the union of all μ-open sets contained in A, that is the largest μ-open set contained in A.

It is easy to observe that c_μ and i_μ are idempotent and monotonic, where $\gamma : \exp X \to \exp X$ is said to be idempotent if and only if $A \subset B \subset X$ implies $\gamma\gamma(A) = \gamma(A)$ and monotonic if and only if $A \subset B \subset X$ implies $\gamma A \subset \gamma B$. It is also well known that from [7, 8] that if μ is a G.T on X and $A \subset X$, $x \in X$ then $x \in c_\mu(A)$ if and only if $x \in M \in \mu \Rightarrow M \cap A \neq \emptyset$ and $c_\mu(X - A) = X - i_\mu(A)$. Let $B \subset \exp X$ and $\emptyset \in B$. Then B is called a base [4] for μ if $\{UB' : B' \subset B\} = \mu$.

Lemma 1.1 [3]. If μ is a generalized topology on *X*, then there is a $\gamma : \exp X \to \exp X$ such that μ is a collection of all γ-open sets. We can suppose that γ satisfies $\gamma\emptyset = \emptyset$, $\gamma A \subset A$, $\gamma\gamma A \subset A$ for $A \subset X$.

In this paper we introduce *N*-closed subset of a generalized topological space and then introduce and study the class of *N*-continuous, μ-*N*-continuous functions between generalized topological spaces. A subset A of a generalized topological space (X, μ) is called *N*-closed (relative to μ) or μ-*N*-closed if for any cover $\mathcal{U}$ of A by μ-open sets, there is a finite sub collection $\mathcal{V}$ of $\mathcal{U}$ such that $A \subset \cup \{i_\mu c_\mu(V) : V \in \mathcal{V}\}$.

The space (X, μ) is nearly μ-compact if and only if X is μ-*N*-closed relative to μ. A function $f : X \to Y$ is called μ-*N*-continuous if for each $x \in X$ and each μ-open set V containing $f(x)$ and having μ-*N*-closed complement there is a μ-open set U containing x such that $f(U) \subset V$.

Functions of course continuous functions stand among the most important and most researched points in every part of mathematics .One purpose of this paper is to emphasize the fact that if the co domain of μ-*N*-continuous function *f* is generalized re topologized in an obvious way then *f* is simply a μ-continuous function. This puts the notion of μ-*N*-continuity in a more natural setting, and the distinction made between the class of μ-continuous mappings and μ-*N*-continuous mappings must be interpreted very strictly.

In addition to the introductory section 1, in section 2, we provide some preliminaries of generalized topology and basic properties of μ-semi regularization topologies. Section 3, deals with co μ *N*-closed generalized topologies. In section 4, we consider the transfer of separation properties between a generalized topological space (X, μ) and its co μ-*N*-closed topology $n(\mu)$. Section 5 deals with products and graph function. The end or the omission of a proof will be denoted by ■.

μ-SEMI-REGULAR TOPOLOGIES

In a generalized topological space (X, μ), a set A is called μ-regular open if $A = i_\mu c_\mu(A)$ and μ-regular closed if $A = c_\mu i_\mu(A)$. Let $r.o_\mu(X)$ denote the collection of all μ-regular open sets in (X, μ). Since the intersection of two μ-regular open sets is μ-regular open, the family of μ-regular open sets forms a base for a smaller generalized topology μ_s on X, called the μ-semi regularization of μ. The space (X, μ) is said to be μ-semi regular if $\mu_s = \mu$. Any regular space is μ-semi regular, but the converse is false.

In one sense this paper is a continuation of my recent study of [6].

Notation: We denote μ-α*A* to denote $i_\mu c_\mu(A)$, often suppressing the μ when there is no confusion possible.

We now give some basic results for μ-semi regularization topologies that we shall require. It is clear that any property which is preserved by enlargement of generalized topology.

Lemma 2.1: If A and B are disjoint μ-open sets in generalized topology (X, μ) then μ-αA and μ-αB are disjoint μ-open sets in (X, μ) containing A and B respectively■.

Lemma 2.2: μ-semi regularization topologies are preserved by generalized topological products.■

The process of μ-semi regularizing a generalized topological space is an idempotent operation. The proof of this depends on the observation that the family of all μ_s-regularly open subsets of (X, μ_s) coincides with the collection of all μ-regularly open subsets of (X, μ).

Lemma 2.3: For any generalized topological space (X, μ) we have, $(\mu_s)_s = \mu_s$■.

Definition 2.1: A generalized topological space X is μ-Hausdorff if for any two distinct points in X has disjoint μ-open sets.

Lemma 2.4 : The generalized topological space X is μ-Hausdorff if and only if (X, μ_s) is μ_s-Hausdorff ■.

A proof of Lemma 2.4 comes immediately from Lemma 2.1 and the general comment about properties preserved by enlargements of generalized topologies.

Definition 2.2: The space X is said to be almost μ-regular if for each μ-regular closed subset A of X and each point $x \in X - A = B$ there are disjoint μ-open sets U and V such that $A \subset U$ and $B \subset V$.

There are almost μ-regular spaces which are not μ-regular and that almost μ-regularity and μ-semi regularity are independent notions.

The following theorem provides the fundamental relationship between the above concepts.

Theorem 2.4: The generalized topological space (X, μ) is almost μ-regular if and only if (X, μ_s) is μ_s-regular.

Proof: Let (X, μ) be almost μ-regular and let C be a μ_s-closed subset of X and $x \in X - C$.

Now $C = \cap \{c_i : i \in I\}$ where c_i is μ-regularly closed set for each $i \in I$.

Then there exist some $j \in I$ such that $x \in X - c_i$. So there are disjoint μ_s-open sets U and V such that $C \subset c_i \subset U$ and $x \in V$. By Lemma 2.1 there are disjoint μ_s-open sets U' and V' such that $C \subset U \subset U'$ and $x \in U \subset U'$. Hence (X, μ_s) is μ_s-regular.

Conversely, let C be a μ-regularly closed set and $x \in X - C$. Since (X, μ_s) is μ-regular, there are disjoint μ_s-open sets U and V such that $C \subset U$ and $x \in V$. Since $\mu_s \subset \mu$, we have (X, μ) is almost μ-regular. ■

Lemma 2.5 : If (X, μ_s) is μ_s-T_0 (resp. μ_s-T_1) then (X, μ) is μ-T_0(resp. μ-T_1), but the converse is false.

Proof : The positive statement follows from the inclusion $\mu_s \subset \mu$. Next let (X, μ) be an infinite set with cofinite G.T. Then (X, μ) is μ-T_1, but (X, μ_s) is the indiscrete generalized topology and so is not even μ_s-T_0 ■.

Lemma 2.6 : (X, μ) is μ-Hausdorff if and only if (X, μ_s) is μ_s-Hausdorff . ■

ON CO-μ-N.CLOSED GENERALIZED TOPOLOGY AND μ-N.CONTINUITY

In this section we study the co-μ-*N*-closed generalized topology *n* (μ) of the strong generalized topology (*X*, μ).

Definition 3.1: A subset *A* of a generalized topological space (*X*, μ) is called μ-*N*. closed (relative to μ) if for any cover α of *A* by μ-open sets there is a finite sub collection $\mathcal{A}$ of such that $A \subset \cup\{ i_\mu c_\mu (V) : V \in \mathcal{A}\}$. The space (*X*, μ) is nearly μ-compact if and only if *X* is μ-*N* closed relative to μ.

For background materials of μ-continuous function [6] may be perused.

Definition 3.2: A function $f : X \to Y$ is called μ-*N* continuous if for each $x \in X$ and each μ-open set *V* containing $f(x)$ and having μ-*N*. closed complement there is an μ-open set *U* containing *x* such that $f(U) \subset V$.

We note that a set *A* is a μ-*N* closed set if *A* possesses property μ-*N* closed. *A* has μ-*N* closed complement if $X - A$ possesses property μ-*N* closed.

Let (*X*, μ) be a generalized topological space, and consider $n'(\mu) = \{U \in \mu : X\text{-}U$ is μ-*N* closed relative to μ$\}$. Since the union of two μ-*N* closed set is μ-*N* closed, we have $n'(\mu)$ is a base for a generalized topology *n* (μ) on *X*, called the co-μ-*N* closed generalized topology of μ on *X*. Note that $n(\mu) \subset \mu$.

The following theorem provides the basic relationship between the generalized topology *n* (μ) and the concept of μ-*N* continuity. The proof is immediate from the definition.

Theorem 3.1: The function $f : (X, \mu_1) \to (Y, \mu_2)$ is μ-*N* continuous if and only if $f : (X, \mu_1) \to (Y, n(\mu_2))$ is μ-continuous. ■

Thus μ-*N* continuity property is μ-continuous in the sense of [6]. Hence all general remarks for μ-continuous properties can be applied to μ-*N* continuous functions.

Definition 3.3 : A GTS (*X*, μ) is said to be

(i) μ-compact [15] if every μ-open cover of *X* has a finite subcover.

(ii) μ_s-Hausdorff if for any two distinct points *x* and *y* in *X*, there exists disjoint μ_s-open sets *U* and *V* such that $x \in U, y \in V$.

Definition 3.4 : For a generalized topological space (*X*, μ),the co μ-compact generalized topology of μ on *X* is denoted by *c* (μ) and is defined by $c(\mu) = \{\emptyset\} \cup \{U \in \mu : X - U$ is μ-compact$\}$. The function $f : X \to Y$ is μ-c-continuous if whenever $U \subset Y$ is an μ-open set with μ-compact complement, $f^{-1}(U)$ is μ-open in *X*.

Lemma 3.2 : If (*X*, *c* (μ) is μ-Hausdorff relative to *c* (μ), then (*X*, μ) is μ-compact.

Proof : Let *x* and *y* be any two distinct points of *X*. There are co μ-compact generalized topology *c* (μ), μ-open sets *U* and *V* such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Hence $X = U \cup (X - U) = (X - V) \cup (X - V)$, so that *X* is the union of two μ-compact sub spaces and hence *X* is μ-compact. ■

Note that a base $c(\mu_s)$ is the collection $\{U \in \mu_s : X - U$ is μ_s-compact$\}$, and that this is a sub collection of $n'(\mu)$. This follows from a sub set *A* of (X, v) is $\mu - N$ closed relative to μ if and only if *A* is μ-compact in (X, μ_s). Thus for any generalized topological space (*X*, μ), we have $c(\mu_s) \subset n(\mu)$. Thus $c(\mu_s) \subset n(\mu) \subset \mu$.

Lemma 3.3 : Let X be a μ-Hausdorff space with strong generalized topology μ.Then n (μ) = c (μ_s).

Proof : As noted above, it is enough to show that n (μ) ⊂ c (μ_s). Since X is μ-Hausdorff, we have (X, μ_s) is μ_s-Hausdorff. Thus every compact μ_s subset of X is μ_s-closed, and hence μ-closed. Therefore n' (μ) ⊂ c (μ_s), so that n (μ) = c(μ_s). ■

Note that Lemma 2.1 requires only that every μ_s-compact set is μ_s-closed, that is that (X, μ_s) be a μ-KC-space, which is condition weaker than μ-Hausdorff.

Corollary 3.4: If (X, μ) is μ-Hausdorff, then c (μ) ⊂ c (μ_s) ⊂ n (μ) .

The following example shows that the μ-Hausdorffness is necessary for Lemma 3.3 and corollary 3.4 .

Example 3.1: Let Y be an infinite set and be a co finite generalized topology on Y. Then c (μ) = n (μ) = μ, while c (μ_s) = μ_s which is the indiscrete generalized topology on Y.

Corollary 3.5: If Y is μ-Hausdorff, then $f : X \rightarrow (Y, \mu)$ is μ-N continuous if and only if $f : X \rightarrow (Y, \mu_s)$ is μ_s-c-continuous.

Remark : If (X, n (μ)) is μ-Hausdorff, so is (X, μ). Thus by Lemma 3.3, n (μ) = c (μ_s) = c (μ) since (X, μ) is μ-semi regular. Thus c (μ) is μ-Hausdorff, so by Lemma 3.2, (X, μ) is μ-compact and hence c (μ) = μ. Hence n (μ) = μ, and so by Theorem 3 .1, $f : X \rightarrow (y, \mu)$ is μ-N continuous if and only if $f : X \rightarrow (Y, \mu)$ is μ-continuous.

Theorem 3.6 : The space (X, n (μ)) is μ-Hausdorff relative to n (μ) if and only if (X, μ) is nearly μ-compact Hausdorff.

Proof : If n (μ) is μ-Hausdorff, then μ is μ-Hausdorff and so μ_s is μ-Hausdorff. By Lemma 3.3, n (μ) = c (μ_s), so that Lemma 3.2 implies μ_s is μ-compact, and hence (X, μ) is nearly μ-compact. Conversely, if (X, μ) is nearly μ-compact and μ-Hausdorff, then (X, μ_s) is μ_s-compact and μ_s-Hausdorff, so that μ_s= c (μ_s). Lemma 3.3 implies that n (μ) = c (μ_s), so that n (μ) is μ-Hausdorff. ■

Proposition 3.7 : If the space (X, μ) is

(1) μ-semi regular then n (μ) = c (μ);

(2) Nearly μ-compact, then c (μ_s) = μ_s ⊂ n (μ);

(3) Nearly μ-compact Hausdorff, then μ_s = n (μ);

(4) μ-compact, then n (μ) = μ.

Proof : (1) Since μ = μ_s, we have n' (μ) = {U ∈ μ : Y – U is μ_s-compact} is also a base for c (μ).

(2) Since μ_s is μ-compact, μ_s = c (μ_s) ⊂ n (μ).

(3) The proof of Theorem 3.6 shows that μ_s = c (μ_s) ⊂ n (μ).

(4) We have the general inclusion c (μ) ⊂ n (μ) ⊂ μ, and c (μ) = μ since μ_s is μ-compact. ■

The following example shows that the μ-semi regularity in Proposition 3.7 (1) is crucial. Let (X, μ) be the half disc topology described as in L.A. Steen and J.A. Seebach "Counter examples in Topology", and μ_e be the usual Euclidean topology on Y. Then μ_s = μ_e , and since μ is μ-Hausdorff by Lemma 3.3, we have n (μ) = c (μ_s) = c (μ_e). To see that c (μ) is a proper subset of n (μ), consider the subset [0, 1] × {0} = B of X. Then B is μ-closed and μ-compact in μ_e so that $X - B \in c\ (\mu_e) = n\ (\mu)$. But B is not μ-compact, since it is a discrete subspace of (X, μ), and hence $X - B \notin c\ (\mu)$.

Proposition 3.7 enables us to obtain conditions on the co domain of a function under which μ-*N*-continuity can be related to existing variations of μ-continuity.

Corollary 3.8 : Let $f : X \rightarrow (y, \mu)$be a function.

(1) If *Y* is μ-semi regular. Then *f* is μ-*N* continuous if and only if *f* is μ-*c* continuous .

(2) If *Y* is nearly μ-compact, then *f* is μ-*N* continuous implies *f* is almost μ-continuous.

(3) If *Y* is nearly μ-compact Hausdorff, then *f* is μ-*N* continuous if and only if *f* is almost μ-continuous.

(4) *Y* is μ-compact, then μ-continuity, μ-*c*-continuous and μ-*N* continuous are equivalent. ■

Example 3.1. Shows that if $X = Y$ an infinite set, with S the indiscrete SGT and μ the co finite topology, then the identity mapping $i : (X, S) \rightarrow (Y, \mu)$ is almost μ-continuous but it is not μ-continuous, but it is not μ-*N*-continuous.

We expect to be able to prove that for any generalized topological space (X, μ) the co-μ-*N* closed topology $n\ (\mu)$ is nearly μ-compact.

Lemma 3.9: If $\mu_s \subset \alpha \subset \mu$, then $\alpha_s = \mu_s$.

Proof : We show that μ-ro (X, α) = μ-ro (X, μ). If F is a μ-closed subset of (X, α), then F is μ-closed in (X, μ) and μ-*int* $F \in$ μ-ro $(X, \mu) \subset \mu_s$. Since μ-*int* $F = \mu_s$-*int* (μ-*int* F) $= \mu_s$-*int* F and μ_s-*int* $F \subset \alpha$-*int* $F \subset$ μ-*int* F, it follows that α-*int* F = μ-*int* F. Thus μ-ro $(X, \alpha) \subset$ μ-ro (X, μ).

Conversely, $G \in$ μ-ro (X, μ) implies that $G \subset \alpha$-*cl* $G \subset \mu_s$-*cl* G = μ-*cl* G. Hence, $G = \mu_s$-*int* $G \subset \alpha$-*int* (α-*cl* G) $\subset$ μ-*int* (μ-*cI* G) $= G$, so that $G = \alpha$-*int* (α-*cl* G) and $G \in$ μ-ro (X, α), thus μ-ro $(X, \mu) \subset$ μ-ro (X, α). ■

Corollary 3.10 : If (X, μ) is nearly μ-compact then $(X, n\ (\mu))$ is nearly μ-compact.

Proof : By proposition 3.7 (2) we have that $\mu_s \subset n\ (\mu) \subset \mu$. By lemma 3.9, $(n\ (\mu))_s = \mu_s$ and so is μ-compact. Thus $n\ (\mu)$ is nearly μ-compact. ■

Lemma 3.11 : If (X, μ) is not nearly μ-compact, then $(n\ (\mu))_s$ is strongly generalized indiscrete.

Proof : Suppose $(n\ (\mu))_s$ is not indiscrete. So there is a $U \in \mu$-ro $(X, n\ (\mu))$ such that $\emptyset \neq U \neq X$. Then $U \in n\ (\mu)$ and $n\ (\mu)$ *cl* $U \neq X$. Let V be a basic $n\ (\mu)$ open set such that $\emptyset \neq V \subset U$. So $V \in \mu$, X-V is μ_s-compact, and there is a basic $n\ (\mu)$ closed sub set F of X such that $V \subset F \neq X$. By definition F is μ_s-compact. Since $X = V \cup (X - V) \subset F \cup (X - V)$, X is μ_s-compact., which contradicts (X, μ) is not nearly μ-compact.■

Corollary 3.12: If (X, μ) is not nearly μ-compact, then $(X, n\ (\mu)\ )$ is nearly μ-compact.■
Corollary 3.10 and 3.12 give the following :

Theorem 3.13 : For any generalized topological space (X, μ), the space $(X, n\ (\mu))$ is nearly μ-compact. ■

If $\mathcal{L}\ (X)$ is the lattice of topologies on a set X, we can regard the process of obtaining $n\ (\mu)$ from μ as an operator $n : \mathcal{L}\ (X) \rightarrow \mathcal{L}\ (X)$. The co compactness operator c is considered in [8] and the co Lindelof operator ℓ in [9]. For any generalized topology μ we have, $c\ (\mu) \subset n\ (\mu) \subset \mu$, so that in particular for $c\ (\mu)$ we obtain $c\ (c\ (\mu)) \subset nc\ (\mu) \subset c\ (\mu)$. Then by proposition (1) of [9], $c\ (c\ (\mu)) = c\ (\mu)$, so that $n\ (c\ (\mu)) = c\ (\mu)$. We use the results of proposition 3.7, for repeated applications of the operators n and *c*. For example, if (*X*, μ) is μ-semi regular then *n* (μ) = *c* (μ), so that *c* (*c* (μ)) = *c* (*n* (μ)) = *n* (*c* (μ)) = *n* (*n* (μ)) = *n* (μ). If (*X*, μ) is μ-Hausdorff, then *n* (μ) = *c* (μ_s), by Lemma 3.3, so that *n* (μ) is μ-compact and *c* (*n* (μ)) = *n* (μ). Proposition 3.7(4) shows that *n* (*n* (μ)) = *n* (μ) so that *c* (*n* (μ)) = *n* (*n* (μ)) = *n* (μ) for any μ-Hausdorff topology.

Example 3.2: Note that the operator does not respect inclusion in $\mathcal{L}$ (*X*). For example, if *X* is the set of all real numbers and μ_1, μ_2, μ_3, μ_4 are the cofinite, usual, Sorgenfrey and discrete strong generalized topologies on *X*, then $\mu_1 \subset \mu_2 \subset \mu_3 \subset \mu_4$. But *n* ($\mu_1$) = *n* ($\mu_4$) is the cofinite generalized topology μ_1, while *n* (μ_2) = *c* (μ_2) properly contains *n* (μ_3) = *c* (μ_3) which properly contains μ_1.

Lemma 3.14 : In the class of μ-Hausdorff strong generalized topologies on a set *X*, the co-μ-compactness operator reverses inclusion, that is if $\mu_1 \subset \mu_2$ then *c* (μ_2) ⊂ *c* (μ_1).

Proof: If *C* is *c* (μ_2) closed, then *C* is μ_2- compact, and hence co μ_1-compact since $\mu_1 \subset \mu_2$. Now μ_1-Hausdorff implies *C* is μ_1-closed so that *C* is *c* (μ_1) closed, and hence *c* (μ_2) ⊂ *c* (μ_1). ■

Corollary 3.15 : The operator *n* reverses inclusion in the class of all μ-Hausdorff semi regular generalized topologies on a set *X*.

Proof : Let $\mu_1 \subset \mu_2$. By proposition 3.7(1), *n* (μ_1) = *c* (μ_1) and *n* (μ_2) = *c* (μ_2). Thus by Lemma 3.14, *n* (μ_2) ⊂ *n* (μ_1). ■

We note that the μ-Hausdorff condition is crucial for the previous results. This is shown by the generalized topologies μ_1 and μ_2 of example 3.2. .

Definition 3.5: A topological space (*X*, μ) is called an μ-R_1-space if every two distinct points with distinct closures have disjoint neighbourhoods.

Separation properties

In this section we consider the transfer of μ-separation properties between a generalized topological space (*X*, μ) and its co-μ-*N* closed generalized topology *n* (μ). Since *n* (μ) ⊂ μ any property preserved by enlargement of generalized topologies will be transferred from *n* (μ) to μ; and μ-T_0, μ-T_1 and μ-Hausdorff, μ-Urysohn, and completely μ-Hausdorff are examples of such μ-separation properties. If (*X*, μ) is an infinite set with the discrete topology in S G T then *n* (μ) is the co finite generalized topology on *X*. Thus the property μ-R_1, μ-Hausdorff, μ-Urysohn, functional μ-Hausdorff, μ-regular, completely μ-regular and μ-normal are not transferred from (*X*, μ) to *n* (μ).

Proposition 4.1: If *X* is μ-T_1, then (*X*, *n* (μ)) is μ-T_1.

Proof: This follows from the fact that any singleton $\{x\}$ of *X* is μ-closed and (nearly) μ-compact in (*X*, μ), and hence is μ-closed in *n* (μ). ■

Proposition 4.2: If *X* is μ-R_0then (*X*, *n* (μ)) is μ-R_0.

Proof : Since (*X*, μ) is μ-R_0, for any $x \in X$ the set μ-*cl* $\{x\}$ is μ-closed and μ-compact, so it is *n* (μ) closed. Hence *n* (μ) *cl* $\{x\}$ ⊂ μ-*cl* $\{x\}$. On the other hand, since *n* (μ) ⊂ μ, we have μ-*cl* $\{x\}$ ⊂ *n* (μ) *cl* $\{x\}$, so that $n\ (\mu)\ cl\ \{x\} = \mu\text{-}cl\ \{x\}$. The result follows from this equality. ■

We have not been able to answer the following questions about the transfer of μ-separation properties.

(1) If (X, μ) is μ-T_0, $(X, n(\mu))$ is μ-T_0.

(2) Are any of the properties of μ-R_0, μ-R_1, (complete) μ-regularity, μ-normality transferred from $(X, n(\mu))$ to (X, μ).

We note in passing that $(X, n(\mu))$ can be μ-compact while (X, μ) is not even μ-locally compact. For example, if (X, μ) is the Sorgenfrey line, then it is μ-semi regular so that $n(\mu) = c(\mu)$ is μ-compact while μ is not μ-locally compact. Proposition 3.7(4) shows that if (X, μ) is μ-compact so is $(X, n(\mu))$, since $n(\mu) = \mu$.

μ-connectedness behaves like μ-compactness. If (X, μ) is μ-connected, so is the smaller GT $n(\mu)$. If (X, μ) is infinite discrete it is not μ-connected (in fact it is μ-totally disconnected), while $(X, n(\mu))$ is a co μ finite space and so is μ-connected.

Products

In this section we consider a collection $\{(X_\alpha, \mu_\alpha)_{\alpha \in \Lambda}\}$ of non empty generalized topological spaces. Let X denote the product $\prod \{Y_\alpha : \alpha \in \Lambda\}$ and μ denote the Tychonoff product topology $\{\prod U_\alpha : \alpha \in \Lambda\}$ on X. We are interested in the relationship between the co-μN closed operator n and generalized topological products.

Lemma 5.1 : The product B of sets B_α is μ-N closed in (X, μ) if and only if each B_α is μ-N closed in (X_α, B_α).

Proof : By Theorem 3.1 of [13], B is μ-N closed in (X, μ) if and only if B is μ-compact in (X, μ_s). Since $\mu_s = \prod (\mu_\alpha)_s$ B is μ_s-compact in (X, μ_s) if and only if B_α is μ-compact in $(X_\alpha, (\mu_\alpha)_s)$ for each $\alpha \in \Lambda$, that is if and only if B_α is μ-N closed in (X_α, μ_α). ■

We now introduce a stronger version of μ-N continuity.

Definition 5.1: A function $f : (X, S) \to (Y, \mu)$ is strongly μ-N continuous if $f : (X, n(S)) \to (Y, n(\mu))$ is μ-continuous.

It is clear from Theorem 3.1 that strong μ-N continuity implies μ-N continuity. On the other hand, strong μ-N continuity is not implicationally related to μ-continuity. For let (X, S) be the set R of real numbers with the discrete topology and (Y, μ) be R with the usual topology. Then $n(S)$ is the co finite generalized topology on R, while $n(\mu) = c(\mu)$, by Proposition 3.7(1).

Note that $n(\mu)$ is strictly larger than $n(S)$. Hence if f is the identity mapping from X to Y, $f : (X, S) \to (Y, \mu)$ is μ-continuous but not strongly μ-N continuous , while $f^{-1} : (Y, \mu) \to (X, S)$ is strongly μ-N continuous but not μ-continuous. Notice that f is μ-N continuous but not strongly μ-N continuous.

Proposition 5.2: If $f : (X, S) \to (Y, \mu)$ is μ-N continuous and $g : (Y, \mu) \to (Z, \alpha)$ is strongly μ-N continuous then $g \circ f : (X, S) \to (Z, \alpha)$ is μ-N continuous. ■

Proposition 5.3: If (X, α) is nearly μ-compact then the projection mapping $P_y : X x\, Y \to Y$ is strongly μ-N continuous, where (Y, μ) is any generalized topological space.

Proof : Let V be any $n(\mu)$ basic open set, so that Y-V is a μ-closed N closed set in (Y, μ) . Then $P_Y^{-1}(V)$ is μ-open in $(X \times Y, S \times \mu)$ since the projection P_Y is μ-continuous. Furthermore, $(X \times Y) - P_Y^{-1}(V) = X \times (Y - V)$ is μ-N closed by Lemma 5.1. ■

Corollary 5.4 : If $\{Y_\alpha : \alpha \in \Lambda\}$ are nearly μ-compact spaces, then the projections $P_\alpha : Y \to Y_\alpha$ are strongly μ-N continuous. ■

Theorem 5.5 : Let (X, S) be nearly μ-compact, and $f : X \to Y$ be a function. If the graph function $g : X \to X \times Y$ is μ-N continuous, then f is μ-N continuous.

Proof : Note that $f = P_Y \circ g$, and appeal to propositions 5.2 and 5.3. ■

Lemma 5.6 : If (Y_α, μ_α) is locally nearly μ-compact and μ-Hausdorff for each $\alpha \in \Lambda$, then $n(\prod \mu_\alpha) \subset \prod n(\mu_\alpha)$.

Proof : For each $\alpha \in \Lambda$, $n(\mu_\alpha) = c((\mu_\alpha)_s)$ by Lemma 3.3 so that $\prod n(\mu_\alpha) = \prod c((\mu_\alpha)_s)$. The product space (Y, μ) is μ-Hausdorff, so that $n(\prod \mu_\alpha) = c((\prod \mu_\alpha)_s)$, again by Lemma 3.3. Now each $(\mu_\alpha)_s$ is locally μ-compact and μ-Hausdorff, so that by Theorem 7 of [8], $c(\prod (\mu_\alpha)_s) \subset \prod c((\mu_\alpha)_s)$. Hence $n(\mu_\alpha) \subset \prod n(\mu_\alpha)$. ■

The following theorem explains the relationship between co μ-compact operator c and generalized topological products.

Theorem 5.7: The equality $\prod c(\mu_\alpha) = c(\prod \mu_\alpha)$ holds if and only if one of the following conditions is satisfied

(1) μ_α is μ-compact, for each $\alpha \in \Lambda$.

(2) $c(\mu_\alpha)$ is indiscrete, for each $\alpha \in \Lambda$.

(3) If for some $k \in \Lambda$, neither μ_k is μ-compact nor $c(\mu_k)$ is indiscrete, then for all $\alpha \in \Lambda - \{k\}$, μ_α is indiscrete.

Proof : In view of Theorem 25 of [8(i)], the inclusion $\prod c(\mu_\alpha) \subset c(\prod(\mu_\alpha))$ ....(*) holds if and only if one of the above three conditions (1) to (3) is satisfied. Thus we only need to show that equality in (*) holds.

(1) If each μ_α is μ-compact then $c(\mu_\alpha) = \mu_\alpha$ for each $\alpha \in \Lambda$, so that $\prod c(\mu_\alpha) = c(\prod \mu_\alpha)$. But $\prod \mu_\alpha$ is μ-compact, so that $\prod \mu_\alpha = c(\prod \mu_\alpha)$.

(2) If each $c(\mu_\alpha)$ is indiscrete, then the product $\prod c(\mu_\alpha)$ is indiscrete. Suppose $c(\prod \mu_\alpha)$ is not discrete, so there is a set $U \in \prod \mu_\alpha$, such that $\emptyset \neq U \neq X$ and $F = X - U$ is μ-compact. Let $x \in F$, say $x = \{< X_\alpha > : \alpha \in \Lambda\}$. Then $c_\mu\{x\} = \prod \mu_\alpha cl\ \{X_\alpha\} \subset F$ and is μ-compact. Hence, for each $\alpha \in \Lambda$, $c_\mu\{x\}$ is μ_α-compact. Since each $c(\mu_\alpha)$ is indiscrete we have $\mu_\alpha\ cl\ \{x\} = X_\alpha$, so that $F = X$ and $U = \emptyset$, a contradiction.

(3) If all μ_α are indiscrete, except at most one μ_k then we have

$$\begin{aligned}\prod c(\mu_\alpha) &= \{P_k^{-1}(U) : U \in c(\mu_k)\}\\ &= \{P_k^{-1}(U) : U \in \mu_k \text{ and } X_k - U \text{ is } \mu_k\text{-compact}\}\\ &= \{P_k^{-1}(U) : U \in \mu_k \text{ and } X - P_k^{-1}(U) \text{ is } \mu\text{-compact}\}\\ &= c(\prod \mu_\alpha).\ ■\end{aligned}$$

Corollary 5.8 : Let (X_α, μ_α) be μ-semi regular for each $\alpha \in \Lambda$. Then $\prod n(\mu_\alpha) = n(\prod \mu_\alpha)$ if and only if one of the conditions (1) – (3) in Theorem 5.7 is satisfied.

Proof : Follows from Proposition 3.7 (1). ■

Corollary 5.9: Let (X_α, μ_α) be μ-Hausdorff non singleton spaces, for each $\alpha \in \Lambda$. Then $\prod n(\mu_\alpha) = n(\prod \mu_\alpha)$ if and only if (X_α, μ_α) is nearly μ-compact for each $\alpha \in \Lambda$.

Proof : Follows from Lemma 3.3. The conditions (1) and (3) do not hold since each $n(\mu_\alpha)$ is μ - T_1 by Proposition 4.1. ■

Proposition 5.10 : If each (X_α, μ_α) is nearly μ-compact, then $\prod n(\mu_\alpha) \subset n(\prod X_\alpha)$.

Proof : For each sub base member P_α^{-1} (U) of the product $\prod n$ (μ_α), where $U \in \mu_\alpha$ and $\alpha \in \Lambda$, $X_\alpha - U$ is μ-*N* closed, so that the set $X - P_\alpha^{-1}$ $(U) = P_\alpha^{-1}$ $(X_\alpha - U)$ is μ-closed and μ-*N* closed, see Lemma 5.1. Thus P_α^{-1} $(U) \in n$ $(\prod \mu_\alpha)$ as required. ■

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