Acta Ciencia Indica, Vol. XLIII M, No. 3, 215 (2017)

VARIATIONAL ITERATION METHOD FOR SOLVING ADVECTION–DISPERSION EQUATIONS

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RECEIVED : 30 November, 2017

In this present paper, variational iteration method is used to solve classical advection – dispersion equation as well asreaction–advection–dispersion equation. It is also applied to advection–dispersion–decay equation which is obtained if there is radioactive tracer having decay rate λ and advection–dispersion equation with growth which is found if there exists biological species tracer having logistic growth rate $F = ru\left(1-\frac{u}{K}\right)$ .Here r is growth constant & K is carrying capacity. The results are found in the form of an infinite series that converges to the closed–form solution.

KEYWORDS: Variational iteration method; advection – dispersion equation.

Introduction

The study of transportation of biological or chemical contaminants through the subsurface aquifer systems is of very much importance in environmental science. Let us take the transport of a biological or chemical tracer which is carried by water through a saturated, unidimensional, uniform porous medium. The standard model for contaminant transport is the advection–dispersion equation. The profile of concentration for a group of particles is determined by this model. The classical unidimensional advection–dispersion equation, $\frac{\partial u}{\partial \tau} = -v\frac{\partial u}{\partial \zeta} + d\frac{\partial^2 u}{\partial \zeta^2}, 0 < \zeta < L \;\&\; \tau > 0$ depicts the tracer plume evolution inserted at $x = 0$ and initially time $t = 0$. Here v is known as advective velocity while d shows the mixed effects of advective dispersion and molecular diffusion [1].

Al Sayed *et al*. [2] used decomposition method [3, 4] of Adomian, for solving atransitional fractional ADE. M. Goyal and G. D. Gupta [5, 6] applied Adomian decomposition method (ADM) in finding solution of three dimensional inhomogeneous differential equations and one dimensional wave equations. J. Duan [7] presented a review of ADM and its applications to fractional differential equations. M. Goyal [8-11] discussed the use of ADM in many applied fields. Dehghan & Shakeri [12] applied variational iteration method (VIM) for finding the solution of the reaction–diffusion equation. M. Goyal [13] used modified decomposition method (MDM) in solving tenth order boundary value problems. He and Sharma [14] solved many linear and nonlinear higher order boundary value problems using MDM.

The aim of this study is to solve the classical ADE and the reaction ADE by using variational iteration method. This recently developed method [15–17] directly attacks the nonlinear partial differential equation without a need to find certain polynomials for nonlinear terms and gives result in an infinite series. It rapidly converges to analytical solution. This method also requires no linearization, discretization or little perturbations. It lessons mathematical computations significantly.

Variational Iteration Method

To solve differential eqn. $Nu(t) + Lu(t) = g(t)$, a correctional functional is constructed as:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda[Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)]\,d\xi$$

where λ is Lagrangian multiplier which may be found optimally with the help of theory of variations. n shows n^{th} order guess, $\tilde{u}_n$ is used as restricted variation *i.e.* $\delta\tilde{u}_n = 0$. Consecutive approximations u_{n+1}, $n \geq 0$, of u are promptly found out using a discriminating function u_0. The exact result is achieved as $u = \lim_{n\to\infty} u_n$.

Numerical Problems

(i) Consider $\dfrac{\partial u}{\partial \tau} = -v\dfrac{\partial u}{\partial \zeta} + d\dfrac{\partial^2 u}{\partial \zeta^2}, 0 < \zeta < L \;\&\; \tau > 0$ … (1)

with IC : $u(\zeta, 0) = e^{-\zeta}$ … (2)

Here, $u(\zeta, \tau)$ = dissolved concentration, v = Darcy velocity and d = dispersion coefficient

We label $x = \dfrac{\zeta}{L}, t = \dfrac{\tau v}{L}$ … (3)

Substituting (3) into Eq. (1), we obtain

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2},\ 0 < x < 1 \text{ and } t > 0 \quad \dots (4)$$

with initial conditions $u(x, 0) = e^{-x}$ … (5)

where $\mu = \dfrac{d}{vL}$ and $P_e = \dfrac{1}{\mu}$ is Peclet number. If the Peclet number is large enough (> 100) then it signals that advection term dominates.

Solution by VIM :

Correctional functional is,

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left[\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial \tilde{u}_n(x,\xi)}{\partial x} - \mu\frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2}\right]d\xi \quad \dots (6)$$

The stationary conditions follow as

$[1 + \lambda]_{\xi = t} = 0$ and $[\lambda']_{\xi = t} = 0$ which quickly gives $\lambda = -1$. … (7)

Putting in Eq. (6),

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} - \mu \frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right] d\xi \quad \text{… (8)}$$

Here, $u_0 = e^{-x}$

$$u_1(x,t) = u_0(x,t) - \int_0^t \left[\frac{\partial u_0(x,\xi)}{\partial \xi} + \frac{\partial u_0(x,\xi)}{\partial x} - \mu \frac{\partial^2 u_0(x,\xi)}{\partial x^2}\right] d\xi = e^{-x}[1 + t(\mu+1)]$$

$$u_2(x,t) = e^{-x}\left[1 + t(\mu+1) + \frac{t^2}{\underline{|2}}(1+\mu)^2\right]$$

Continuing like this, we get,

$$u = \sum_{n=0}^{\infty} u_n(x,t) = e^{-x} e^{t(1+\mu)} = e^{-x+t(1+\mu)} \quad \text{… (9)}$$

(ii) Consider the reaction ADE:

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \frac{F}{\omega}, 0 < x < 1 \quad \text{and} \quad t > 0 \quad \text{… (10)}$$

where ω = porosity of medium such that $0 < \omega < 1$ and F is source term.

(a) If there exists a radioactive tracer with rate of decay λ, then $F = -\lambda \omega u$ & we get the linear advection–dispersion–decay Eq. [1]

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} - \lambda u, 0 < x < 1 \quad \& \quad t > 0 \quad \text{… (11)}$$

Correctional functional is,

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left[\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial \tilde{u}_n(x,\xi)}{\partial x} - \mu \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} + \lambda \tilde{u}_n(x,\xi)\right] d\xi \quad \text{…(12)}$$

Again, we obtain $\lambda = -1$. … (13)

Substituting in Eq. (12),

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} - \mu \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \lambda u_n(x,\xi)\right] d\xi \quad \text{…(14)}$$

Here, $u_0 = e^{-x}$

$$u_1(x,t) = u_0(x,t) - \int_0^t \left[\frac{\partial u_0(x,\xi)}{\partial \xi} + \frac{\partial u_0(x,\xi)}{\partial x} - \mu \frac{\partial^2 u_0(x,\xi)}{\partial x^2} + \lambda u_n(x,\xi)\right] d\xi = e^{-x}[1 + t(1+\mu-\lambda)]$$

$$u_2(x,t) = e^{-x}[1 + t(1+\mu-\lambda) + \frac{t^2}{\underline{|2}}(1+\mu-\lambda)^2]$$

Continuing like this, we get,

$$u = \sum_{n=0}^{\infty} u_n(x,t) = e^{-x} e^{t(1+\mu-\lambda)} = e^{-x+t(1+\mu-\lambda)} \qquad \ldots (15)$$

(b) If there exists biological species tracer having growth rate which is logistic and given by $F = ru\left(1 - \frac{u}{K}\right)$, we get ADE with growth [1] as

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \frac{r}{\omega} u \left(1 - \frac{u}{K}\right), 0 < x < 1 \ \& \ t > 0 \qquad \ldots (16)$$

Correctional functional is,

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi) \left[\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial \tilde{u}_n(x,\xi)}{\partial x} - \mu \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} - \lambda_1 \tilde{u}_n(x,\xi) - \lambda_2 \tilde{u}_n^2(x,\xi) \right] d\xi \qquad \ldots (17)$$

Here $\qquad \lambda_1 = \dfrac{r}{\omega}$ and $\lambda_2 = -\dfrac{r}{\omega K} \qquad \ldots (18)$

Again, $\qquad \lambda = -1. \qquad \ldots (19)$

Substituting in the Eq. (17),

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,\xi)}{\partial \xi} + \frac{\partial u_n(x,\xi)}{\partial x} - \mu \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - \lambda_1 u_n(x,\xi) - \lambda_2 u^2{}_n(x,\xi) \right] d\xi \qquad \ldots (20)$$

Here, $u_0 = e^{-x}$

$$u_1(x,t) = u_0(x,t) - \int_0^t \left[\frac{\partial u_0(x,\xi)}{\partial \xi} + \frac{\partial u_0(x,\xi)}{\partial x} - \mu \frac{\partial^2 u_0(x,\xi)}{\partial x^2} - \lambda_1 u_0(x,\xi) - \lambda_2 u_0{}^2(x,\xi) \right] d\xi$$

$$= e^{-x}[1 + t(1+\mu+\lambda_1)] + \lambda_2 t e^{-2x}$$

$$u_2(x,t) = e^{-x}[1 + t(1+\mu+\lambda_1) + \frac{t^2}{\underline{|2}}(1+\mu+\lambda_1)^2] + \lambda_2 t e^{-2x} + \lambda_2 t^2 e^{-2x}[2 + 3\mu + \frac{3}{2}\lambda_1 + \lambda_2 e^{-x}]$$
$$+ \frac{t^3}{3}\lambda_2 e^{-2x}[(1+\mu+\lambda_1)^2 + \lambda_2{}^2 e^{-2x} + 2\lambda_2 e^{-x}(1+\mu+\lambda_1)]$$

$$u_2(x,t) = e^{-x}[1 + t(1+\mu+\lambda_1) + \frac{t^2}{\underline{|2}}(1+\mu+\lambda_1)^2] - \frac{\lambda_1}{K} e^{-2x}[t + t^2(2 + 3\mu + \frac{3}{2}\lambda_1) - \frac{\lambda_1}{K} t^2 e^{-x}$$
$$- \frac{t^3}{3}(1+\mu+\lambda_1 - \frac{\lambda_1}{K} e^{-x})^2] \qquad \Big| \text{ Putting } \lambda_2 = -\frac{\lambda_1}{K}$$

Continuing like this, we get,

$$u = \sum_{n=0}^{\infty} u_n(x,t) \qquad \ldots (21)$$

Conclusion

In this work, He's VIM is successfully applied to find out the solution of classical advection–dispersion equation and the fundamental reaction–advection–dispersion equation with cases of tracer. It is found that the results are in the form of infinite series which easily and quickly converge to the exact solution. Variational iteration method has a much simpler and elegant approach particularly in solving nonlinear equations.

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