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CONSTRUCTION OF GROUP DIVISIBLE DESIGNS FROM GENERALIZED ROW ORTHOGONAL CONSTANT COLUMN MATRICES

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Some series of Group Divisible (GD) designs have been constructed from Generalized Orthogonal Constant Column Matrices (GROCM). Some GD designs listed in Clatworthy's Table have been constructed from these series.

KEYWORDS : Balanced Incomplete Block Design, GD design, Circulant Matrix, GROCM.

INTRODUCTION

We recall following definitions:

1.1. BALANCED INCOMPLETE BLOCK DESIGN (BIBD):

Let $V = \{1, 2, 3, \ldots, v\}$ be a non-empty set and $\beta = \{\beta_1, \beta_2, \ldots, \beta_b\}$ be a multiset of subsets of V. The elements of V are called treatments and the elements of β are called blocks. A BIBD is an arrangement of v treatments into b blocks such that each block contains k treatments, each treatment belongs to r blocks and each pair of treatments belongs to λ blocks. v, b, r, k, λ are called parameters of the BIBD. These parameters are not all independent but are related by the following relations:

(i) $vr = bk$ (ii) $r(k-1) = \lambda(v-1)$.

A BIBD for which $v = b$ (hence $r = k$) is called a Symmetric BIBD (SBIBD).

1.2. Circulant Matrix:

An $n \times n$ matrix $C = [c_{ij}]_{0 \le i,j \le n-1}$ where $c_{ij} = c_{j-i(m0dn)}$ is a circulant matrix of order n.

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix} = circ(c_0, c_1, \ldots, c_{n-1})$$

1.3. *m*-Class Association Scheme (AS):

Let X be a non-empty set of order v. A set $\Omega = \{R_0, R_1, \ldots, R_m\}$ of non-empty relations on X is an m-class AS if following properties are satisfied

(i) $R_0 = \{(x, x)\}: x \in X\}$

(ii) Ω is a partition of $X \times X$ *i.e.*

$$\bigcup_{i=0}^{m} R_i = X \times X, R_i \cap R_j = \phi \text{ if } I \neq j.$$

(iii) $R_i^T = R_i \text{ } where \text{ } R_i^T = \{(x, y): (y, x) \in R_i\}, i = 0, 1, \ldots, m.$

(iv) Let $(x, y) \in R_i$. For $i, j, k \in 0, 1, \ldots, m$

$$p_{jk}^i = p_{kj}^i = \left|\{z: (x, z) \in R_j \bigcap (z, y) \in R_k\}\right|$$

which is independent of $(x, y) \in R_i$.

If $(x, y) \in R_i$ then *x* and *y* are called *i*-th associates. For a given treatment $\alpha \in X$, the number of treatments which are *i*-th associates of α is n_i ($i = 0, 1, 2, \ldots, m$), where the number n_i is independent of the treatment α chosen. The non-negative integers *v*, n_i, $p_{jk}^i(i, k, j = 0, 1, \ldots, m)$ are called the parameters of the *m*-Class AS. Every treatment is zero-th associate of itself. These parameters are not all independent but are connected by the following relations

(i) $\sum_{i=1}^{m} n_i = v - 1$

(ii) $\sum_{k=1}^{m} p_{jk}^i = \begin{cases} n_j - 1 \text{ } if \text{ } i = j \\ n_j \text{ } if \text{ } i \neq j \end{cases}$

(iii) $n_i p_{jk}^i = n_j p_{ik}^j$

For details see Godsil and Song [7].

1.4. Association Matrices-

These matrices were introduced by Bose and Mesner [2].

The i-th association matrix $B_i = [b_{\alpha\beta}^i]_{\substack{0 \le i \le m \\ \alpha, \beta \in X}}$ of an m-class AS is a symmetric matrix of order *v* where

$$b_{\alpha\beta}^i = \begin{cases} 1 \text{ if } \alpha \text{ and } \beta \text{ are mutually ith associates} \\ 0 \text{ Otherwise} \end{cases}$$

1.4.1. Properties of Association Matrices:

(i) $B_0 = I_v$ (ii) $\sum_{i=0}^{m} B_i = J_v$ (iii) $B_i B_j = B_j B_i = \sum_{k=0}^{m} p_{ij}^k \text{ } (i, j = 0, 1, 2, \ldots, m)$

1.5. Partially Balanced Incomplete Block (PBIB) Design:

Let *X* be a non-empty set with cardinality *v*. The elements of *X* are called treatments. A PBIB design based on an m-class association scheme is a family of *b* subsets (blocks) of *X*, each of size *k* such that each treatment occurs in *r* blocks, any two treatments occur together in λ_i ($i = 0, 1, \ldots, m$) blocks if they are mutually *i*th associates. *v*, *b*, *r*, *k*, λ_i are called parameters of a PBIB design.[5]

The following relations connect these parameters of PBIB design and also of the parent association scheme:

(i) $vr = bk$

(ii) $\sum_{i=0}^{m} n_i \lambda_i = rk$, where $\lambda_0 = r$.

1.6. Group divisible (GD) AS :

A GD AS is an arrangement of $v = mn$ treatments in a rectangular array of *m* rows and *n* columns such that any two treatments belonging to the same row are first associates and remaining pairs of treatments are second associates.

The parameters of GD AS are as follows:

$$v = mn,\ n_1 = n - 1,\ n_2 = n\ (m - 1)$$

$$P_1 = \begin{bmatrix} n-2 & 0 \\ 0 & n(m-1) \end{bmatrix},\ P_2 = \begin{bmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{bmatrix}$$

1.7. Group divisible design (GDD):

A GDD is a 2-class PBIB design based on a GD AS of $v = mn$ treatments arranged with b blocks such that each block contains k distinct treatments, each treatment occurs in exactly r blocks and any two treatments which are first associates occur together in λ_1 blocks, whereas any two treatments which are second associates occur together in λ_2 blocks. v, b, r, k, λ_1, λ_2 are called the parameters of the GDD.

Let N be the incidence matrix of a GD design with parameters $v = mn$, b, r, k, λ_1, λ_2. Then the eigenvalues (θ_i) and the corresponding multiplicities

(α_i) of the matrix NN^T are given by

$$\theta_0 = rk,\ \theta_1 = r - \lambda_1,\quad \theta_2 = rk - v\lambda_2,\ \alpha_0 = 1,\ \alpha_1 = m\ (n - 1),\ \alpha_2 = m - 1.$$

GD designs have been classified into following three categories based on the eigenvalues of NN^T

(i) Singular, if $r = \lambda_1$

(ii) Semiregular, if $r > \lambda_1$ and $rk = v\lambda_2$

(iii) Regular, if $r > \lambda_1$ and $rk > v\lambda_2$.

GD designs havebeen studied by Bose and Connor [1], Dey [4], Dey and Nigam [6], John and Turner [8], Rao [9], Seberry [10] and so on. GDDs are suitable for factorial experiment [5].

For convenience, I_n denotes the identity matrix of order n, $J_{t\times u}$ denotes the $t \times u$ matrix all of whose entries are 0, $K_{t\times u} = J_{t\times u} - I_{t\times u}$. $e_{t\times 1}$ denotes the $t \times 1$ matrix with all its entries 1 and Γ_i is a square matrix of order p whose ith column has all entries 1 and remaining columns have entries in 0. $A \otimes B$denotes Kronecker product of matrices A and B.$\alpha^i = \text{circ.}(0, 0, 0, \dots 1, \dots, 0)$ is a circulant matrix of order n with 1 at (i + 1)-th position such that $\alpha^n = I_n$.

GROCM AND ITS REDUCTION TO INCIDENCE MATRIX OF A GDD

2.1 Definition of GROCM

Singh and Prasad [11] defined Generalized Orthogonal Combinatorial matrix (GOCM). Here we define GROCM.

Let $N = [N_{ij}]$, $i, j \in \{1, 2, \dots, m\}$ where N_{ij} are {0, 1} matrices oforder $n \times s_j$. Let $R_i = (N_{i1}, N_{i2}, \dots, N_{im})$ be the ith row of blocks. We define inner product of two rows of blocks R_i and R_j as $R_i \circ R_j = R_i R_j^T = \sum_{k=1}^{m} N_{ik} N_{jk}^T$.

N is called a Generalized Row Orthogonal Matrices (GROM) if there exists fixed positive integer r and fixed non-negative integers λ_1, λ_2, λ_3 such that

$$R_i \circ R_j = R_i R_j^T = \sum_{k=1}^{m} N_{ik} N_{jk}^T = \begin{cases} rI_n + \lambda_1 K_n\ if\ i = j \\ \lambda_2 I_n + \lambda_3 K_n\ if\ i \neq j. \end{cases}$$

A {0, 1}-matrix N is called a constant column matrix if sum of entries in each column of N is constant. A GROM with constant column sum k will be called GROCM.

$v = mn, b = m(s_1 + s_2 + \cdots + s_m), r, k, \lambda_1, \lambda_2, \lambda_3, m, n$ are called the parameters of the GROCM. $\lambda_1, \lambda_2, \lambda_3$ are called concurrences of the GROCM.

Theorem 2.2: A GROCM N with concurrences λ_1, λ_2, λ_3 is an incidence matrix of a GD design if either $\lambda_1 = \lambda_3$ or $\lambda_2 = \lambda_3$.

Proof:

$$NN^T = \begin{pmatrix} rI_n + \lambda_1 K_n & \cdots & \lambda_2 I_n + \lambda_3 K_n \\ \vdots & \ddots & \vdots \\ \lambda_2 I_n + \lambda_3 K_n & \cdots & rI_n + \lambda_1 K_n \end{pmatrix}$$

$$= r(I_m \otimes I_n) + \lambda_1 (I_m \otimes K_n) + \lambda_2 (K_m \otimes I_n) + \lambda_1 (K_m \otimes K_n).$$

Let $\lambda_2 = \lambda_3$. Let

$$B_0 = I_m \otimes I_n, B_1 = I_m \otimes K_n, B_2 = K_m \otimes I_n + K_m \otimes K_n = K_m \otimes J_n$$

Then

$$B_0^2 = I_{mn}, B_1^2 = (n-1)B_0 + (n-2)B_1,$$

$$B_1^2 = n(m-1)(B_0 + B_1) + n(m-2)B_2, B_1 B_2 = B_2 B_1 = (n-1)B_2$$

Thus B_0, B_1, B_2 are association matrices of a GD AS. Hence N is the incidence matrix of a GDD based on a GD AS represented by the array

$$\begin{array}{cccc} 1 & 2 & \ldots & n \\ n+1 & n+2 & \ldots & 2n \\ \vdots & \vdots & \vdots & \vdots \\ (m-1)n+1 & (m-1)n+2 & \ldots & mn \end{array}.$$

Let $\lambda_1 = \lambda_3$.We denote λ_2 by λ_1 and $\lambda_1 (= \lambda_3)$ by λ_2.

Then $B_0 = I_m \otimes I_n, B_1 = K_m \otimes I_n, B_2 = I_m \otimes K_n + K_m \otimes K_n = J_m \otimes K_n$

$$B_1^2 = (m-1)B_0 + (m-2)B_1, B_2^2 = m(n-1)(B_0 + B_1) + m(n-2)B_2,$$

$$B_1 B_2 = B_2 B_1 = (m-1)B_2$$

Thus B_0, B_1, B_2 are association matrices of a GD AS. Hence N is the incidence matrix of a GDD based on the GD AS represented bytranspose of the array

$$\begin{array}{cccc} 1 & 2 & \ldots & n \\ n+1 & n+2 & \ldots & 2n \\ \vdots & \vdots & \vdots & \vdots \\ (m-1)n+1 & (m-1)n+2 & \ldots & mn \end{array}$$

Block kronecker product of two block matrices

Let $M = [M_{ij}]_{\substack{i=1,2,\ldots,l \\ j=1,2,\ldots,m}}$ and $N = [N_{pq}]_{\substack{p=1,2,\ldots,r \\ q=1,2,\ldots,s}}$ are two block matrices where M_{ij} and N_{pq} are square matrices of same order.

We define Block Kronecker Product $M \circledast N$ of M and N as

$$M \circledast N = \begin{bmatrix} M_{11}[N_{pq}] & M_{12}[N_{pq}] & \cdots & M_{1m}[N_{pq}] \\ M_{21}[N_{pq}] & M_{22}[N_{pq}] & \ldots & M_{2m}[N_{pq}] \\ \vdots & \vdots & \vdots & \vdots \\ M_{(l-1)1}[N_{pq}] & M_{(l-1)2}[N_{pq}] & \ldots & M_{(l-1)m}[N_{pq}] \\ M_{l1}[N_{pq}] & M_{l2}[N_{pq}] & \cdots & M_{lm}[N_{pq}] \end{bmatrix}$$

Where $$M_{11}[N_{pq}] = [M_{11}N_{pq}] = \begin{bmatrix} M_{11}N_{11} & M_{11}N_{12} & \cdots & M_{11}N_{1q} \\ M_{11}N_{21} & M_{11}N_{22} & \cdots & M_{11}N_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ M_{11}N_{(p-1)1} & M_{11}N_{(p-1)2} & \cdots & M_{11}N_{(p-1)q} \\ M_{11}N_{p1} & M_{11}N_{p2} & \cdots & M_{11}N_{pq} \end{bmatrix}$$

$M_{11}N_{11}$ denotes usual matrix multiplication of M_{11} and N_{11}.

CONSTRUCTION THEOREMS

Theorem 4.1: There exists a GDD with parameters

$v = p^2$, $b = p\ (s + \mu p)$, $r = s + \mu p$, $k = p$, $\lambda_1 = s$, $\lambda_2 = \mu$, $m = n = p$, where μ, s, are non-negative integers and p is a prime.

Proof : Let $N_1 = \mu$ copies of block matrix $\begin{bmatrix} I_p \\ I_p \\ \vdots \\ I_p \end{bmatrix}$

$$N_2 = s \text{ copies of block matrix } \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_p \end{bmatrix}$$

and $$N_3 = \mu \text{ copies of block matrix } \begin{bmatrix} I_p & I_p & \cdots & I_p \\ \alpha & \alpha^2 & \cdots & \alpha^{p-1} \\ \alpha^2 & (\alpha^2)^2 & \cdots & (\alpha^{p-1})^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha^{p-1} & (\alpha^2)^{p-1} & \cdots & (\alpha^{p-1})^{p-1} \end{bmatrix}$$

where $\alpha = \text{circ.}(0, 1, 0, \ldots, 0, 0)$ is a circulant matrix of order p with 1 at 2nd position such that $\alpha^p = I_p$. Then $N = [N_1\ N_2\ N_3]$ is the incidence matrix of the GDD with the required parameters. [vide theorem 2.2]

Corollary 4.1.1: There exists a semiregular GDD with parameters

$v = p^2$, $b = \mu p^2$, $r = \mu p$, $k = p$, $\lambda_1 = 0$, $\lambda_2 = \mu$, $m = p - 1$, $n = p$ where μ is a positive integer.

Proof : On putting $s = 0$ in Theorem 4.1 we get a regular GDD with the required parameters. [vide theorem 2.2]

Corollary 4.1.2 : There exists a semiregular GDD with parameters

$v = p^2 - lp$, $b = \mu p^2$, $r = \mu p$, $k = p - 1$, $\lambda_1 = o$, $\lambda_2 = \mu$, $m = p - 1$, $n = p$ where μ is a positive integer.

Proof : On removing l rows of blocks from incidence matrix of the previous design we obtain a semiregular GDD with the required parameters. [vide theorem 2.2]

Theorem 4.2 : There exists a semiregular GDD with parameters

$$v = b = p^3, r = k = p^2, \lambda_1 = 0, \lambda_2 = p, m = p^2, n = p.$$

Proof : $N = N_1 \circledast N_2$ is an incidence matrix of a semiregular GDD with the required parameters where

$$N_1 = N_2 = \begin{bmatrix} I & I & I & \cdots & I \\ I & \alpha & \alpha^2 & \dots & \alpha^{p-1} \\ I & \alpha^2 & (\alpha^2)^2 & \dots & (\alpha^{p-1})^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I & \alpha^{p-1} & (\alpha^2)^{p-1} & \dots & (\alpha^{p-1})^{p-1} \end{bmatrix}$$

[vide Theorem 2.2]

Corollary 4.2.1 : There exists a semiregular GDD with parameters

$$v = p^3 - lp, b = p^3, r = p^2, k = p^2 - l, \lambda_1 = 0, \lambda_2 = p, m = p^2 - l, n = p.$$

Proof : On removingl rows of blocks from N, we obtain a semiregular GDD with the required parameters.

Theorem 4.3 : The existence of a SBIBD with parameters $v = b = 4n - 1$, $r = k = 2n - 1$, $\lambda = n$ implies the existence of a GDD with parameters

$$v = 2\,(4n - 1), b = 4n, r = 2n, k = 4n - 1, \lambda_1 = 0, \lambda_2 = n, m = 4n - 1, n = 2.$$

Proof : Let N_1 be the incidence matrix of a $(4n - 1, 2n - 1, n)$-design. Then

$N = \begin{pmatrix} N_1 & e_{(4n-1)\times 1} \\ J - N_1 & 0_{(4n-1)\times 1} \end{pmatrix}$ is an incidence matrix of a GDD with the required parameters. [Vide theorem 2.2]

Table of Designs

Table of Some GD designs in the range $2 \le r, k \le 10$ constructed from present theorems which are listed in Clatworthy's Table. [3]

Table I

List of GD designs in Clatwarthy's Table	Source
*R*1, *R*2, *R*3, *R*4, *R*5, *R*6, *R*7, *R*8, *R*9, *R*10, *R*11, *R*12, *R*13, *R*14, *R*15, *R*16, *R*17, *R*59, *R*60, *R*61, *R*62, *R*63, *R*64, *R*65, *R*66, *R*67, *R*68, *R*155, *R*156, *R*157, *R*158, *R*184, *R*185	Theorem 4.1
*SR*1, *SR*2, *SR*3, *SR*4, *SR*5, *SR*23, *SR*24, *SR*25, *SR*60, *SR*6I, *SR*87	Cor.4.1.1
*SR*6, *SR*7, *SR*8, *SR*11, *SR*12, *SR*14, *SR*28, *SR*31, *SR*46, *SR*48, *SR*62, *SR*77	Cor.4.1.2
*SR*102	Theorem 4.2
*SR*25, *SR*43, *SR*56, *SR*73, *SR*85, *SR*94	Cor.4.2.1
*SR*80	Theorem 4.3

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