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CONSTRUCTION OF SOME COMBINATORIAL DESIGNS FROM GENERALIZED ROW ORTHOGONAL CONSTANT COLUMN MATRICES

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A series of L_2-type and Balanced incomplete block designs have been constructed from Generalized Row Orthogonal Constant Column Matrices (GROCM). Some L_2-type designs given in Clatworthy's table have been constructed using these series.

MSC : 05B

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INTRODUCTION

We recall following definitions:

1.1 BALANCED INCOMPLETE BLOCK DESIGN (BIBD)

Let $V = \{1, 2, 3, \ldots, v\}$ be a non-empty set and $\beta = \{\beta_1, \beta_2, \ldots, \beta_b\}$ be a multiset of subsets of V. Elements of V are called treatments and those of β are called blocks. A BIBD is an arrangement of v treatments into b blocks of V such that

(i) Each block contains k treatments.

(ii) Each treatment belongs to r blocks.

(iii) Each pair of treatments belongs to λ blocks.

v, b, r, k, λ are called parameters of the BIBD.

These parameters are not all independent but are related by the following relations:

(i) $vr = bk$ (ii) $r(k-1) = \lambda(v-1)$.

A BIBD for which $v = b$ (hence $r = k$) is called a Symmetric BIBD (SBIBD).

1.2. Circulant Matrix

An $n \times n$ matrix $C = [c_{ij}]_{0\le i,j\le n-1}$ where $c_{ij} = c_{j-i(m0dn)}$ is called a circulant matrix of order n.

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix} = cir\ c(c_0, c_1, \ldots, c_{n-1})$$

1.3. LATIN SQUARE AND MUTUALLY ORTHOGONAL LATIN SQUARES (MOLS)

A Latin square L of order n is an $n \times n$ array based on a set of n symbols such that every row and every column of L contains every symbol exactly once.

Let $A=\left(a_{ij}\right)_{i,j=1,2,\ldots,n}$ and $B=\left(b_{ij}\right)_{i,j=1,2,\ldots,n}$ be two Latin squares of order n based on symbol sets $N = \{a_1, a_2, \ldots, a_n\}$ and $M = \{b_1, b_2, \ldots, b_n\}$ respectively. Then A and B are said to be MOLS if their symbolic product $(A, B) = \left[(a_{ij}, b_{ij}\right]_{i,j=1,2,\ldots,n}$ which is an nxn array of n^2 ordered pairs, has all its entries distinct.

1.4. ASSOCIATION SCHEME (AS)

Let X be a non-empty set of order v. A set $\Omega = \{R_0, R_1, \ldots, R_d\}$ of non-empty relations on X is a d-class AS if following properties are satisfied

(i) $R_0 = \{(x, x)\}: x \in X\}$

(ii) Ω is a partition of $X \times X$ *i.e.*

$$\cup_{i=0}^{d} R_i = X \times X, R_i \cap R_j = \phi \text{ if } i \neq j.$$

(iii) $R_i^T = R_i$ where $R_i^T = \{(x, y): (y, x) \in R_i\}, i = 0,1, \ldots, d.$

(iv) Let $(x, y) \in R_i$. For $i, j, k \in 0,1, \ldots, d$

$$p_{jk}^i = p_{kj}^i = \left|\{z: (x, z) \in R_j \bigcap (z, y) \in R_k\}\right|$$

which is independent of $(x, y) \in R_i$.

If $(x, y) \in R_i$ then x and y are called ith associates. For a given treatment $\alpha \in X$, the number of treatments which are i-th associates of α is n_i (i = 0, 1, 2, …, d), where the number n_i is independent of the treatment α chosen. The non-negative integers v, n_i, p_{jk}^i(i, k, j = 0, 1, …, d) are called the parameters of the d-Class AS. Every treatment is zero-th associate of itself [6].

These parameters are not all independent but are connected by the following relations

(i) $\sum_{i=1}^{d} n_i = v - 1$

(ii) $\sum_{k=1}^{d} p_{jk}^i = \begin{cases} n_j - 1 \text{ } if \text{ } i = j \\ n_j \text{ } if \text{ } i \neq j \end{cases}$

(iii) $n_i p_{jk}^i = n_j p_{ik}^j$

1.5.1. Association Matrices

These matrices were introduced by Bose and Mesner [2].

Consider an d-class AS. The i-th association matrix $B_i = [b_{\alpha\beta}^i]_{\substack{0 \le i \le d \\ \alpha,\beta \in X}}$ of a d-class AS is a symmetric matrix of order v where

$$b^i_{\alpha\beta} = \begin{cases} 1 \text{ if } \alpha \text{ and } \beta \text{ are mutually } i\text{th associates} \\ 0 \text{ otherwise} \end{cases}$$

1.5.2 Properties of Association Matrices

(i) $B_0 = I_v$ (ii) $\sum_{i=0}^{d} B_i = J_v$ (iii) $B_i B_j = B_j B_i = \sum_{k=0}^{m} p_{ij}^k \; (i, j = 0,1,2, \ldots, d)$

1.6. Partially Balanced Incomplete Block (PBIB) Design

Let X be non-empty set with cardinality v. A PBIB design based on a d-class association scheme is a family of b subsets (blocks) of X, each of size k such that each treatment occurs in r blocks, any two treatments occur together in λ_i (i = 0, 1, …, d) blocks if they are mutually i-th associates. v, b, r, k, λ_i are called parameters of the PBIB design.

The following relations connect these parameters of PBIB design and also of the parent association scheme:

(i) $vr = bk$

(ii) $\sum_{i=0}^{d} n_i \lambda_i = rk$, where $\lambda_0 = r$.

1.7. Latin square AS (L_i AS)

Let $v = s^2$ treatments be arranged in a square array of s rows and s columns. We assume that $i - 2$ ($i \geq 2$) MOLS of side s exist which are superimposed on the square array. On these s^2 treatments we define L_i AS as follows:

Two distinct treatments are first associates if they occur in the same row or same column of the array or in positions occupied by the same symbol in any of the Latin squares otherwise any two distinct treatments are second associates.

The parameters of the L_iAS ($i \geq 2$) are

$$v = s^2, n_1 = i(s-1), n_2 = (s-1)(s-i+1),$$

$$P_1 = (p_{ij}^1) = \begin{bmatrix} (i-1)(i-2)+s-2 & (i-1)(s-i+1) \\ (i-1)(s-i+1) & (s-i)(s-i+1) \end{bmatrix},$$

$$P_2 = (p_{ij}^2) = \begin{bmatrix} i(i-1) & i(s-1) \\ i(s-i) & (s-i)(s-i-1)+s-2 \end{bmatrix}$$

1.8. L_i-Type design

A PBIB design based on the L_i AS is called a L_i-Type design. Let N be the incidence matrix of a GD design with parameters $v = mn$, b, r, k, λ_1, λ_2. Then the eigen values (θ_i) and the corresponding multiplicities (α_i) of the matrix NN^T are given by

$$\theta_0 = rk,\ \theta_1 = r + (s-i)\lambda_1 - (s-i+1)\lambda_2,\ \theta_2 = r - i\lambda_1 + (i-1)\lambda_2,$$

$$\alpha_0 = 1,\ \alpha_1 = i(s-1),\ \alpha_2 = (s-i+1)(s-1).$$

In this paper we consider only L_2-Type designs.

BIBDs have been studied by Bose [1], Kageyama [8], Rao [10], Shrikhande and Raghvarao [11], Yalavigi [13] and so on. L_i-Type designs have been studied by Clayworthy [3], Mesner [9] and so on. L_i-Type designs are suitable for factorial experiment. [5]

For convenience, I_n denotes the identity matrix of order n, $J_{t\times u}$ denotes the $t \times u$ matrix all of whose entries are 1, $K_{t\times u} = J_{t\times u} - I_{t\times u}$. $e_{t\times 1}$ denotes the $t \times 1$ matrix with all its entries 1 and Γ_i is a square matrix of order p whose ith column has all entries 1 and remaining columns have entries 0. $A \otimes B$ denotes Kronecker product of matrices A and B. $\alpha^i = \text{circ.}(0,0,0, \ldots 1, \ldots, 0)$ is a circulant matrix of order n with 1 at (i + 1)-th position such that $\alpha^n = I_n$.

GROCM AND ITS REDUCTION TO AN INCIDENCE MATRIX OF A BIBD AND L2-TYPE DESIGNS

2.1 Definition of GROCM

Singh and Prasad [12] defined Generalized Orthogonal Combinatorial matrix (GOCM). Here we define GROCM.

Let $N = [N_{ij}]$, $i, j \in \{1, 2, ..., m\}$ where N_{ij} are $\{0, 1\}$-matrices oforder $n \times s_j$. Let $R_i = (N_{i1}, N_{i2}, ..., N_{im})$ be the ith row of blocks. We define inner product of two rows of blocks R_i and R_j as $R_i \circ R_j = R_i R_j^T = \sum_{k=1}^{m} N_{ik} N_{jk}^T$.

N is called a Generalized Row Orthogonal Matrices (GROM) if there exists fixed positive integer r and fixed non-negative integers $\lambda_1, \lambda_2, \lambda_3$ such that

$$R_i \circ R_j = R_i R_j^T = \sum_{k=1}^{m} N_{ik} N_{jk}^T = \begin{cases} rI_n + \lambda_1 K_n \text{ if } i = j \\ \lambda_2 I_n + \lambda_3 K_n \text{ if } i \neq j. \end{cases}$$

A $\{0, 1\}$-matrix *N* is called a constant column matrix if sum of entries in each column of N is constant. A GROM with constant column sum *k* will be called GROCM.

$v = mn, b = m(s_1 + s_2 + \cdots + s_m), r, k, \lambda_1, \lambda_2, \lambda_3$ are the parameters of the GROCM. $\lambda_1, \lambda_2, \lambda_3$ are called concurrences of the GROCM.

Theorem 2.2. A GROCM with concurrences $\lambda_1, \lambda_2, \lambda_3$ is in an incidence matrix of

(i) An L_2-type PBIB design if $m = n = s$ (let) and $\lambda_1 = \lambda_2$

(ii) A BIBD if $\lambda_1 = \lambda_2 = \lambda_3$

Proof : (i) Let *N* be a GROCM. Let $m = n = s$. Then

$$NN^T = \begin{pmatrix} rI_s + \lambda_1 K_s & \cdots & \lambda_2 I_s + \lambda_3 K_s \\ \vdots & \ddots & \vdots \\ \lambda_2 I_s + \lambda_3 K_s & \cdots & rI_s + \lambda_1 K_s \end{pmatrix}$$

$$= r(I_s \otimes I_s) + \lambda_1 (I_s \otimes K_s) + \lambda_2 (K_s \otimes I_s) + \lambda_1 (K_s \otimes K_s).$$

Let $\lambda_1 = \lambda_2$. Then let

$$B_0 = I_s \otimes I_s B_1 = I_s \otimes K_s + K_s \otimes I_s, B_3 = K_s \otimes K_s$$

We have

$$\sum_{i=0}^{2} B_i = J_{s^2}$$

$$B_1^2 = 2(s-1)B_0 + (s-2)B_1 + 2B_2$$

$$B_2^2 = (s-1)^2 B_0 + (s-1)(s-2)B_1 + (s-2)^2 B_2$$

$$B_1 B_2 = (s-1)B_1 + 2(s-2)B_2$$

Thus B_0, B_1, B_2 are association matrices of an L_2 *AS*. Hence *N* is the incidence matrix of an L_2-type PBIB design based on L_2 *AS* which is arranged as

$$\begin{matrix} 1 & 2 & \dots & s \\ s+1 & s+2 & \dots & 2s \\ \vdots & \vdots & \vdots & \vdots \\ (s-1)s+1 & (s-1)s+2 & \dots & s^2 \end{matrix}$$

(ii) Let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. Then

$$NN^T = rI_n + \lambda K_n \text{ and } kN = kJ_n$$

Hence in this case, N is an incidence matrix of a BIBD with parameters

$$v = mn, b = m(s_1 + s_2 + \dots + s_m), r, k, \lambda.$$

CONSTRUCTION THEOREMS

Theorem 3.1 There exists a SBIBD with parameters

$$v = p^2, b = sp(p+1), r = s(p+1), k = p, \lambda = s.$$

Proof : Let $N_1 = s$ copies of block matrix $\begin{bmatrix} I_p \\ I_p \\ \vdots \\ I_p \end{bmatrix}$

$$N_2 = s \text{ copies of block matrix } \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_p \end{bmatrix}$$

and $$N_3 = s \text{ copies of block matrix } \begin{bmatrix} I_p & I_p & \dots & I_p \\ \alpha & \alpha^2 & \dots & \alpha^{p-1} \\ \alpha^2 & (\alpha^2)^2 & \dots & (\alpha^{p-1})^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha^{p-1} & (\alpha^2)^{p-1} & \dots & (\alpha^{p-1})^{p-1} \end{bmatrix}$$

where $\alpha = \text{circ.}(0, 1, 0, \dots, 0, 0)$ is a circulant matrix of order p with 1 at 2nd position such that $\alpha^p = I_p$.

Then, Then $N = [N_1\ N_2\ N_3]$ is the incidence matrix of a BIBD with the required parameters. [vide Theorem 2.2]

Particular cases are

H. No. 2, 11, 24.[7]

Theorem 3.2 There exists an L_2-type design with parameters

$$v = p^2, b = p[2s + t(p-1)], r = 2s + t(p-1), k = p, \lambda_1 = s, \lambda_2 = t, n_1 = 2(p-1)$$
$$n_2 = (p-1)^2$$

Proof : Let $N_1 = s$ copies of block matrix $\begin{bmatrix} I_p \\ I_p \\ \vdots \\ I_p \end{bmatrix}$

$$N_2 = s \text{ copies of block matrix } \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_p \end{bmatrix}$$

and
$$\mathrm{N}_3 = \text{t copies of block matrix } \begin{bmatrix} I_p & I_p & \dots & I_p \\ \alpha & \alpha^2 & \dots & \alpha^{p-1} \\ \alpha^2 & (\alpha^2)^2 & \dots & (\alpha^{p-1})^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha^{p-1} & (\alpha^2)^{p-1} & \dots & (\alpha^{p-1})^{p-1} \end{bmatrix}$$

where $\alpha = \text{circ.}(0, 1, 0, \dots, 0, 0)$ is a circulant matrix of order *p* with 1 at 2nd position such that $\alpha^p = I_p$.

Then $N = [N_1\ N_2\ N_3]$ is the incidence matrix of a L_2-type design with the required parameters. [vide Theorem 2.2]

Particular cases are

*LS*12, *LS*13, *LS*14, *LS*15, *LS*62, *LS*63, *LS*66, *LS*97. [4]

Corollary 3.2.1: There exists a L_2-type design with parameters

$$v = p^2, b = t(p-1)p, r = t(p-1), k = p, \lambda_1 = 0, \lambda_2 = t, n_1 = 2(p-1), n_2 = (p-1)^2$$

Proof : On putting *s* = *o* in the previous theorem we obtain a L_2-type design with the required parameters.[vide theorem 2.2]

Particular cases are

*LS*61, *LS*64. [4]

Corollary 3.2.2. There exists a L_2-type design with parameters

$$v = p^2, b = 2ps, r = 2s, k = p, \lambda_1 = s, \lambda_2 = 0, n_1 = 2(p-1), n_2 = (p-1)^2$$

Proof : On putting *t* = 0 in the theorem 3.2 we obtain a L_2-type design with the required parameters.[vide theorem 2.2]

Particular cases are

*LS*7, *LS*8, *LS*9, *LS*10, *LS*11, *LS*51, *LS*54, *LS*57, *LS*59, *LS*60, *LS*84, *LS*89, *LS*95, *LS*96. [4]

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