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BE-ALGEBRAS WITH ZERO ELEMENT

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The concept of BE-Algebra has been introduced by H.S. Kim and Y.H. Kim in 2006. Since then different other concepts have been developed by several authors. In general, a BE-Algebra may not contain a zero element. But a BE algebra with zero element has some different significance. Here we study such BE-Algebras.

KEYWORDS : BE-Algebra, commutative BE-Algebra, Clopen set.

INTRODUCTION

Definition (1.1): Let $(X; *, 1)$ be a system of type (2, 0) consisting of a non-empty set X, a binary operation $*$ and a fixed element 1. The system $(X; *, 1)$ is called BE-Algebra if the following conditions are satisfied:

(BE 1) $x * x = 1$

(BE 2) $x * 1 = 1$

(BE 3) $1 * x = x$

(BE 4) $x * (y * z) = y * (x * z)$

for all $x, y, z \in X$.

Example (1.2) : Let $X = \{0, 1\}$ and let the binary operation '*' be defined as the Cayley table

Table (1)

*	0	1
0	1	1
1	0	1

Then $(X; *, 1)$ is a BE-Algebra.

Note (1.3) : Here 1 can be replaced by any a.

Example (1.4): Let $Y = X^3 = \{(x_1, x_2, x_3) : x_i = 0 \text{ or } 1\}$

Let $0 \equiv (0\ 0\ 0)$, $1 \equiv (0, 0, 1)$ $2 \equiv (0, 1, 0)$ $3 \equiv (0, 1, 1)$

$4 \equiv (1, 0, 0)$ $5 \equiv (1, 0, 1)$ $6 \equiv (1, 1, 0)$ $7 \equiv (1, 1, 1)$

We extend binary operation '.' in Y from table (1). Then Cayley table for $Y = \{0, 1, 2, 3, 4, 5, 6, 7\}$ is given by

Table (2)

.	0	1	2	3	4	5	6	7
0	7	7	7	7	7	7	7	7
1	6	7	6	7	6	7	6	7
2	5	5	7	7	5	5	7	7
3	4	5	6	7	4	5	6	7
4	3	3	3	3	7	7	7	7
5	2	3	2	3	7	7	6	7
6	1	1	3	3	5	5	7	7
7	0	1	2	3	4	5	6	7

Definition (1.5) : Let $(X; *, 1)$ be a BE-Algebra. If X contains an element $0 \in X$ such that $0 * x = 1$ for all $x \in X$, then X is called a BE-Algebra with zero element 0.

Definition (1.6) : In a BE-algebra $(X, *, 1)$, a binary operation "+" is defined as

$$x + y = (x * y) * y$$

A BE-Algebra $(X, *, 1)$ is said to be commutative iff $x + y = y + x$ for all $x, y \in X$

Definition(1.7) : Let $(X; *, 1)$ be a BE-Algebra with zero element 0 such that 0 commutes with each $x \in X$, *i.e*, $(0 * x) * x = (x * 0) * 0$, then the complement of x, denoted as x^c, is defined as $x^c = x * 0$. Example 1 and example 2 satisfies conditions of above definitions.

Lemma (1.8) : We have $(x^c)^c = x$ for every $x \in X$.

Proof : Let $x^c = y$, then

$$y^c = y * 0 = (x * 0) * 0 = (0 * x) * x = 1 * x = x$$

Hence the result.

Definition (1.9) : A BE-Algebra $(X, *, 1)$ is said to be

(a) Self-distributive, if for any $x, y, z \in X$

$$x * (y * z) = (x * y) * (x * z)$$

(b) Transitive, if for any x, $y, z \in X$

$$(y * z) \leq (x * y) * (x * z), \text{ i.e., } (y * z) * ((x * y) * (x * z)) = 1$$

Definition (1.10) : A non-empty subset A of a BE-algebra X is called an essence of X if $X * A = A$.

MAIN RESULTS

Theorem (2.1) : A non-empty collection T of subsets of a given set X is a BE-Algebra with zero element $0 = \Phi$ and unit element X in which 0 commutes with every $A \in T$ iff T is closed with respect to complement and union.

Proof : Let T be a BE-Algebra with binary operation '*', zero element $0 \equiv \Phi$ and unit element $1 \equiv X$.

If $T = \{\Phi, X\}$ then the condition is satisfied as $\Phi^c = X$, $X^c = \Phi$ and $\Phi \cup X = X \in T$.

Let $\Phi \neq A \neq X$ be an element of T. From the given condition Φ commutes with A, so $A^c = A * \Phi \in T$. So T is closed with respect to complement.

Since T is a BE-Algebra, $A * A = X$ => either $A * A = A^c \cup A$ or $A * A = A \cup A^c$. This means that for $A, B \in T$, The binary operation '*' is defined either as

$$A * B = A^c \cup B \text{ or } A * B = A \cup B^c,$$

But the condition $A * X = X$ is satisfied only for $A * B = A^c \cup B$. …(A_1)

Also, in this case $X * A = X^c \cup A = A$ is satisfied.

So our binary operation * is given as $A * B = A^c \cup B$.

Also condition (BE 4) is satisfied for binary operation '*' given by (A_1).

Let $A, B \in T$, then $A^c, B \in T$ which gives, $A^c * B = A \cup B \in T$. This proves that T is closed with respect to union.

Conversely, suppose that T is closed with respect to complement and union.

For A, $B \in T$, we define

$$A * B = A^c \cup B \qquad \dots (A_2)$$

Then '*' is a binary operation in T.

Also, for $A \in T$, we have

(BE 1) $A * A = A^c \cup A = X \equiv 1$;

(BE 2) $A * X = A^c \cup X = X \equiv 1$;

(BE 3) $X * A = \Phi \cup A = A$;

(BE 4) Let $A, B, C \in T$, then

$$\begin{aligned} A * (B * C) &= A * (B^c \cup C) \\ &= A^c \cup (B^c \cup C) \\ &= (A^c \cup B^c) \cup C \\ &= (B^c \cup A^c) \cup C \\ &= B^c \cup (A^c \cup C) \\ &= B^c \cup (A * C) \\ &= B * (A * C) \end{aligned}$$

Since $\Phi * A = X \cup A = X$, Φ is a zero element of X.

This proves that (T; *, $1 \equiv X$) is a BE-Algebra with zero element $\Phi \equiv 0 \in T$.

Also $(\Phi * A) * A = 1 * A = A$ and $(A * \Phi) * \Phi = A^c * \Phi = A$

So Φ commutes with each $A \in T$.

Corollary (2.2) : T is closed with respect to intersection.

Corollary (2.3) : If T is finite then T defines a Topology on X. The elements of T are clopen sets.

Corollary (2.4) : If X is a topological space and T be the collection of clopoen subsets of X, then T is a BE-Algebra with respect to binary operation defined by (A_2).

Example (2.5) : Let $X = \{a, b, c, d, e\}$ and let $T = \{\Phi, A, B, C, D, E, F, X\}$, where $A = \{a, b\}$, $B = \{a, b, c\}$, $C = \{c\}$, $D = \{c, d, e\}$, $E = \{d, e\}$, $F = \{a, b, d, e\}$. Then the Cayley table for binary operation '*' defined by (A_2) is given as

Table (3)

*	1	*A*	*B*	*C*	*D*	*E*	*F*	0
1	1	*A*	*B*	*C*	*D*	*E*	*F*	0
A	1	1	1	*D*	*D*	*D*	1	*D*
B	1	*F*	1	*D*	*D*	*E*	*F*	*E*
C	1	*F*	1	1	1	*F*	*F*	*F*
D	1	*A*	*B*	*B*	1	*F*	*F*	*A*
E	1	*B*	*B*	*B*	1	1	1	B
F	1	*B*	*B*	*C*	*D*	*D*	1	*C*
0	1	1	1	1	1	1	1	1

where $0 \equiv \Phi$ and $1 \equiv X$. Here $(T; *, 1)$ is a BE-Algebra with zero element 0. Also T is a topological space in which each element is clopen set.

Example (2.6) : Let $X = \{a, b, c\}$ and let $T = P(X)$. Let $0 \equiv \Phi$, $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, $D = \{b, c\}$, $E = \{a, c\}$, $F = \{a, b\}$, $1 \equiv X$. Then binary operation '*' defined by (A) is given by the Cayley table

Table (4)

*	1	*A*	*B*	*C*	*D*	*E*	*F*	0
1	1	*A*	*B*	*C*	*D*	*E*	*F*	0
A	1	1	*D*	*E*	*D*	1	1	*D*
B	1	*E*	1	*E*	1	*E*	1	*E*
C	1	*F*	*F*	1	1	1	*F*	*F*
D	1	*A*	*F*	*E*	1	*E*	*F*	*A*
E	1	*F*	*B*	*D*	*D*	1	F	B
F	1	*E*	*D*	*C*	*D*	*E*	1	*C*
0	1	1	1	1	1	1	1	1

Then $(T; *, 1)$ is a BE-Algebra with zero element 0.

Also essences of X are $\{1, D\}$, $\{1, E\}$, $\{1, F\}$ and $\{1, A, E, F\}$.

Theorem (2.7) : The BE-Algebra $(T; *, X)$ considered in theorem (2.1) is

(i) commutative

(ii) self-distributive

(iii) transitive.

Proof : (i) For $A, B \in T$ we have

$$(A * B) * B = (A^c \cup B) * B = (A \cap B^c) \cup B$$

$$= (A \cup B) \cap (B^c \cup B)$$

$$= (A \cup B) \cap X = (A \cup B)$$

Similarly, $(B * A) * A = B \cup A$.

Since, $A \cup B = B \cup A$, we have $(A * B) * B = (B * A) * A$.

Hence, $(T; *, X)$ is commutative.

(ii) For $A, B, C \in T$, we have

$$\begin{aligned} A * (B * C) &= A * (B^c \cup C) \\ &= A^c \cup (B^c \cup C) \\ &= (A^c \cup B^c) \cup C \\ &= (A \cap B)^c \cup C \end{aligned}$$

Also,

$$\begin{aligned} (A * B) * (A * C) &= (A^c \cup B) * (A^c \cup C) \\ &= (A^c \cup B)^c \cup (A^c \cup C) \\ &= (A \cap B^c) \cup (A^c \cup C) \\ &= (A \cup (A^c \cup C)) \cap (B^c \cup (A^c \cup C)) \\ &= ((A \cup A^c) \cup C) \cap ((B^c \cup A^c) \cup C) \\ &= (X \cup C) \cap ((B^c \cup A^c) \cup C) \\ &= X \cap ((A^c \cup B^c) \cup C) \\ &= ((A^c \cup B^c) \cup C) \\ &= (A \cap B)^c \cup C \end{aligned}$$

So, $(T; *, X)$ is self-distributive.

(iii) For $A, B, C \in T$, we have,

$$\begin{aligned} &(B * C) * ((A * B) * (A * C)) \\ &= (B * C) * (A * (B * C)) \\ &= (B^c \cup C) * ((A^c \cup (B^c \cup C)) \\ &= (B^c \cup C) * ((A^c \cup B^c) \cup C) \\ &= (B \cap C^c) \cup ((A^c \cup B^c) \cup C) \\ &= \{B \cup ((A^c \cup B^c) \cup C)\} \cap \{C^c \cup ((A^c \cup B^c) \cup C)\} = X \cap X = X \end{aligned}$$

So, $(T; *, X)$ is Transitive.

Hence the result.

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