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Q*D-SETS IN TOPOLOGICAL SPACE

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In the year 2010, the concepts of Q^* closed sets were introduced and studied by Murugalingam and Lalitha [13, 14] in topological spaces. The set of all Q^*-closed (resp. Q^*-open) sets with X (resp. ϕ) is a topology. Here, we introduce and investigate Q^*D - sets by using the notion of Q^* -open sets to obtain some weak separation axioms.

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Introduction

Separation axioms are one among the most common, important and interesting concepts in Topology. It can be used to define more restricted classes of topological spaces. The separation axioms of topological spaces are usually denoted with the letter "T" after the German "Trennung" which means separation. Most weak separation axioms are defined in terms of generalized closed sets and their definitions are deceptively simple. However, the structure and the properties of those spaces are not always that easy to comprehend. The separation axioms that were studied together in this way were the axioms for Hausdorff spaces, regular spaces and normal spaces. Separation axioms and closed sets in topological spaces have been very useful in the study of certain objects in digital topology [7, 8] Khalimsky, Kopperman and Meyer [9] proved that the digital line is a typical example of $T_{\frac{1}{2}}$ - spaces. There were many definitions offered, some of which assumed some separation axioms before the current general definition of a topological space. For example, the definition given by Felix Hausdorff in 1914 is equivalent to the modern definition plus the Hausdorff separation axiom. The development of the separation axioms between T_0 and T_1 stared with the work of Young in 1943 [19]. In recent years many separation axioms and generalizations of closed sets have been developed by various authors. Throughout this paper, (X, τ), (Y, σ), (Z, μ) (or simply X, Y, Z) denote topological spaces. The purpose of this paper is to introduce a new separation axiom Q^*D_1 which is strictly between $Q^* - T_0$ and $Q^* - T_1$.

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Preliminaries

Definition 2.1. A subset A of a topological space (X, τ) is called

(i) a Q^*-closed if int $(A) = \phi$ and A is closed.

(ii) a Q^*-open if cl $(A) = X$ and A is open.

Definition 2.2. A subset A of a topological space (X, τ) is called :

(a) *D*-set [18] if there are two open sets U and V such that $U \neq X$ and $A = U - V$.

(b) *sD*-set [2] if there are two semi-open sets U and V such that $U \neq X$ and $A = U - V$.

(c) *bD*-set [6] if there are two *b*-open sets U and V such that $U \neq X$ and $A = U - V$.

(d) *pD*-set [11] if there are two pre - open sets U and V such that $U \neq X$ and $A = U - V$.

(e) *gD*-set [1] if there are two *g*-open sets U and V such that $U \neq X$ and $A = U - V$.

(f) *πgb-D*-set [17] if there are two *πgb*-open sets U and V such that $U \neq X$ and $A = U - V$.

Proposition 2.1. (a) Every *D*-set is an α*D*-set,

(b) Every α*D*-set is an *sD*-set, and

(c) Every α*D*-set is a *pD*-set.

Remark 2.1. The set of all Q^*-closed (resp. Q^*-open) sets with X (resp. ϕ) is a topology. It is denoted by σ^*.

Q*D-SET

Tong [18] introduced the notion of *D*-sets by using open sets and used this notion to define some separation axioms. In this section we introduce and study the concept of *Q*D* -set in toplogical spaces.

Definition 3.1. A subset A of a topological space X is called a *Q*D*-set if there are two sets $U, V \in Q^*O(X, \tau)$ such that $U \neq X$ and $A = U \setminus V$.

Clearly *Q**-open set need not be *Q*D*-set.

Example 3.1. Let $X = \{a, b, c, d\}$ and let $\tau = \{\phi, X, \{a\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$. Clearly, $Q^*O(X, \tau) = \{X, \{a\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$. Then $\{b\}$ is a *Q*D*-set.

Theorem 3.1. Every *Q*D*-set is *D*-set, *αD*-set, *pD*-set, *bD*-set, *sD*-set, *gD*-set.

Converse of the above statement need not be true as shown in the following example.

Example 3.2. Let $X = \{a, b, c\}$ and let $\tau = \{\phi, X, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$. Then $\{b\}$ is a *bD*-set but it is not a *Q*D*-set.

Example 3.3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Clearly, $\pi GBO(X, \tau) = \{\phi, \{a\}, \{b\}, \{b, c\}, \{a, c\}, \{a, b\}, X\}$. Then $\{c\}$ is a *πgb-D*-set but not *Q*D*-set.

Example 3.4. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{c\}\}$. Clearly, $PO(X, \tau) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\}$. Then $\{b\}$ is *pD*-set but not *Q*D*-set.

Example 3.5. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$. Since $\alpha O(X, \tau) = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then $\{b\}$ is a α*D*-set but not *Q*D*-set.

Example 3.6. In example 3.1, $SO(X, \tau) = \{\phi, X, \{a\}, \{a, d\}, \{a, b\}, \{a, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c\}, X\}$. Then $\{b, c\}$ is a *sD*-set but not *Q*D*-set.

Example 3.7. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{c\}, \{a, c\}\}$. Clearly, $gO(X, \tau) = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$. Then $\{c\}$ is a gD-set but not Q^*D-set.

Example 3.8. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, c\}\}$. Then $\{c\}$ is D-set but not Q^*D-set.

Remark 3.3. From the above, we have the following diagram of implications

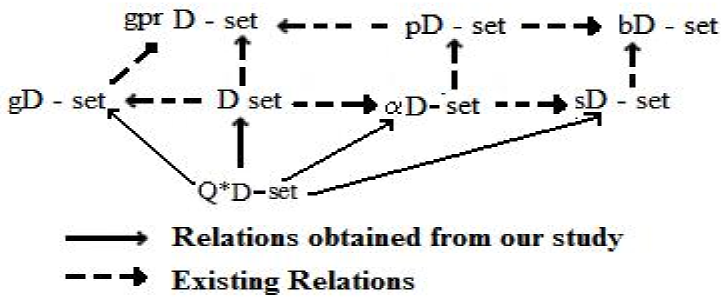

Fig. 3.1. Relationship between existing D sets and new type of D sets in topological spaces

Definition 3.2. A topological space (X, τ) is called Q^* - D_0 if for any distinct pair of points x and y of X there exists a Q^*D-set of X containing x but not y or a Q^*D-set of X containing y but not x.

Definition 3.3. A topological space (X, τ) is called Q^* - D_1 if for any distinct pair of points x and y of X, there exists a Q^*D-set of X containing x but not y and a Q^*D-set of X containing y but not x.

The following example shows that semi - D_1 space does not imply Q^* - D_1 space as well as D_1 - space in the sense of J.Tong [18].

Example 3.9. If $X = \{a, b, c, d\}$ and let $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ be the topology on X. Then X is a semi D_1 - space which is not Q^* - D_1 as well as D_1.

Definition 3.4. A topological space (X, τ) is called Q^* - D_2 if for any distinct pair of points x and y of X there exist disjoint Q^*D-sets G and E of X containing x and y, respectively.

Remark 3.4.

1. If (X, τ) is Q^* - T_i, then (X, τ) is T_i, $i = 0, 1, 2$ and the converse is false.

2. If (X, τ) is Q^* - T_i, then it is Q^* - T_{i-1}, $i = 1, 2$ and the converse is false.

3. If (X, τ) is Q^* - D_i, then it is Q^* - D_{i-1}, $i = 1, 2$.

Theorem 3.2. [8] In any space X, the following hold:

(1) An arbitrary intersection of Q^*-closed sets is Q^*-closed.

(2) The finite union of Q^*-closed sets is Q^*-closed.

Theorem 3.3. For a topological space (X, τ), X is Q^* - D_1 if and only if it is Q^* - D_2.

Proof. Sufficiency. Follows from remark 3.4.

Necessity. Suppose that X is Q^* - D_1. Then for each distinct pair $x, y \in X$, we have Q^*D - sets G_1, G_2 such that $x \in G_1$, $y \notin G_1$, $y \in G_2$ and $x \notin G_2$. Let $G_1 = U_1 / U_2$, $G_2 = U_3 / U_4$, where $U_1, U_2, U_3, U_4 \in Q^*O(X, \tau)$ and $U_1 \neq X$, $U_3 \neq X$. By $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. Now we consider two cases

(1) $x \notin U_3$. By $y \notin G_1$ we have two subcases

(a) y $\notin U_1$. By $x \in U_1 / U_2$, it follows that $x \in U_1 / (U_2 \cup U_3)$ and by $y \in U_3/U_4$ we have $y \in U_3 / (U_2 \cup U_4)$. Also, $((U_1 / (U_2 \cup U_3) \cap (U_3 / (U_1 \cup U_4)) = \phi$. Observe also from 3.6.2 (1) that $U_2 \cup U_3$ and $U_1 \cup U_4$ are Q*-open.

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 / U_2$, $y \in U_2$ and $(U_1 / U_2) \cap U_2 = \phi$. Observe that $U_2 \neq X$ since $G_1 \neq \phi$, thus by Remark 3.1, U_2 is a Q*D-set.

(2) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 / U_4$, $x \in U_4$ and $(U_3 / U_4) \cap U_4 = \phi$. Observe that $U_4 \neq X$ since $G_2 \neq \phi$, thus by Remark 3.1, U_4 is a Q*D-set. Hence, X is Q* - D_2.

Definition 3.5. A point $x \in X$ which has X as the only Q*-neighborhood is called a Q* - neat point.

Theorem 3.4. For a Q*-T_0 topological space (X, τ) whose cardinality is greater than 1, the following are equivalent:

1. (X, τ) is Q* - D_1.
2. (X, τ) has no Q*-neat point.

Proof. (1) → (2) : Since (X, τ) is Q* - D_1, each point x of X is contained in a Q*D-set $O = U / V$ and thus in U. By definition $U \neq X$. This implies that x is not a Q*-neat point.

(2) → (1) : Since X is Q* - T_0, so for each $x \neq y$, $\exists$ *a Q*-open* set U containing x but not y. *i.e.* $U \neq X$ and $y \notin U$ we have X has no Q*-neat points so $\exists$ a Q*-open set $V \neq X$ *s.t.* $y \in V$. Hence $y \in V / U$ and $x \notin V / U$. Hence, X is Q*- D_1.

Theorem 3.5. For a Q*-T_1 topological space (X, τ) whose cardinality is greater than 1, the following are equivalent:

1. (X, τ) is Q* - D_1.
2. (X, τ) has no Q*-neat point.

Remark 3.5. It is clear that a Q* - T_0 topological space (X, τ) is not Q*- D_1 if and only if there is a unique Q*- neat point in X. It is unique because if x and y are both Q*-neat points in X, then at least one of them say x has a Q*-neighborhood U containing x but not y. But this is a contradiction since $U \neq X$.

Remark 3.6. In any space which is Q*- D_1 that is not Q* - T_1 is Q*- D_2 that is not Q*- T_2.

Example 3.10. Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, \{a\}, \{a, c\}, X\}$. Then (X, τ) is Q* - T_0 since is T_0 but not Q* - D_1 since there is not a Q*D-set containing c but not a.

The following example shows that Q* - D_1 space need not be Q* - T_1.

Example 3.11. Let $X = \{a, b, c, d\}$ with topology $\tau = \{\phi, \{a\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}, X\}$. Clearly Q*O $(X, \tau) = \{X, \{a\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$. Hence (X, τ) is Q* - D_1 space but not Q* - T_1 since each Q* - open set containing b contains a.

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be Q*-irresolute if the inverse image of every Q* open set in Y is Q* open in X.

Theorem 3.5. If $f : (X, \tau) \to (Y, \underline{\sigma})$ is a Q*-irresolute surjective function and E is a Q*D - set in Y, then the inverse image of E is a Q*D - set in X.

Proof. Let E be a Q*D - set in Y. Then there are Q*- open sets U_1 and U_2 in Y such that $E = U_1 \setminus U_2$ and $U_1 \neq Y$. By the Q*-irresoluteness of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are Q* - open in X. Since $U_1 \neq Y$ and f is surjective, $f^{-1}(U_1) \neq X$. Hence, $f^{-1}(E) = f^{-1}(U_1)/f^{-1}(U_2)$ is a Q*D-set.

Theorem 3.6. If (Y, σ) is Q^*- D_1 and $f : (X, \tau) \rightarrow (Y, \sigma)$ is Q^*- irresolute and bijective then (X, τ) is Q^* - D_1.

Proof. Suppose that Y is a Q^*- D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is Q^*- D_1, it follows from Theorem 3.2 (2) that Y is Q^*- D_2, thus there exist disjoint Q^*D - sets G_x and G_y of Y containing $f(x)$ and $f(y)$, respectively. By Theorem 3.5, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint Q^*D - sets in X containing x and y, respectively. Hence, X is Q^*- D_1.

Theorem 3.7. A topological space (X, τ) is Q^*- D_1 if and only if for each pair of distinct points $x, y \in X$, there exists a Q^* - irresolute surjective function $f : (X, \tau) \rightarrow (Y, \sigma)$, where (Y, σ) is a Q^* - D_1 space such that $f(x)$ and $f(y)$ are distinct.

Proof. Necessity. For every pair of distinct points of X, it suffices to take the identity function on X.

Sufficiency. Let x and y be any pair of distinct points in X. By hypothesis, there exists a Q^* - irresolute function f from (X, τ) onto a Q^* - D_1 space (Y, σ) such that $f(x) \neq f(y)$. Thus by Theorem 3.6.2 (2), (Y, σ) is Q^*- D_2, and therefore there exists disjoint Q^*D - sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is Q^* - irresolute and surjective, by Theorem 3.5, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint Q^*D - sets in X containing x and y respectively. Hence, X is Q^* - D_1.

Theorem 3.8. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a Q^*-continuous surjective function and S is a D-set of (Y, σ), then the inverse image of S is a Q^*D-set of (X, τ).

Proof. Let U_1 and U_2 be two open sets of (Y, σ). Let $S = U_1 - U_2$ be a D-set and $U_1 \neq Y$. We have $f^{-1}(U_1) \in Q^*O\,(X, \tau)$ and $f^{-1}(U_2) \in Q^*O\,(X, \tau)$ and $f^{-1}(U_1) \neq X$. Hence $f^{-1}(S) = f^{-1}(U_1 - U_2) = f^{-1}(U_1) - f^{-1}(U_2)$. Hence $f^{-1}(S)$ is a Q^*D-set.

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