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CONSTRUCTION OF BLOCK STRUCTURED COMPLEX HADAMARD MATRICES

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Some infinite series of complex Hadamard matrices have been discussed in which each 2 × 2 block is a complex Hadamard matrix. Along with it two infinite series of half-full weighing matrix are constructed whose 4 × 4 blocks are also half-full weighing matrices.

KEYWORDS : Hadamard matrix, complex Hadamard matrix, Williamson matrix, Williamson type matrices.

Mathematical Subject Class : 05B20.

Introduction

In the recent years block structured Hadamard matrices have been sought, first by S. S. Agaian [2] and his results were proved from different approaches by Singh and Topno [6]. Agaian names them Block circulant Hadamard (BCH) matrices. Furthermore he provided new optimal codes from given optimal codes using BCH matrices [1]. We rather call them block-structured Hadamard matrices, since we provide block Hadamard matrices which are not circulant block-wise here. Such matrices are generalized to orthogonal designs and employed to construct anticirculant structured block weighing matrices [7] in case of real Hadamard matrices. Such constructions naturally motivate the results to be generalized to complex Hadamard matrices. In the construction of Agaian and Singh *et. al* results have been proved for Hadamard matrices having 4 × 4 blocks, since it is conjectured that an Hadamard matrices of order $4n$, exist for all $n \in N$. In such matrices, 2 × 2 matrices are of different interest. Craigen conjectured for 2 × 2 block structured Hadamard matrices differently [3]. Our approach and purpose are different. Yet since it has been conjectured that complex Hadamard matrices exist for all even orders, we generalize above results for 2 × 2 block structured complex Hadamard matrices. In this note we discuss some of the block structured complex Hadamard matrices, existence of some are quite known, though undocumented. Furthermore some new series have been introduced.

Preliminary

Hadamard matrices (H-matrices) arose as the extremal solutions of the maximum determinant problem where entries a_{ij} are such that $| a_{ij} | \leq 1$. An **Hadamard matrix** is a $(1, -1)$-matrix H of order n such that $HH^T = nI_n$. Throughout this paper I stands for identity matrix, order of which should be determined by the context. A **complex Hadamard matrix** of order n is its generalization where the entries are from the set $\{1, -1, i, -i\}$ such that

$HH^* = nI_n$. In this matrix also $|a_{ij}| = 1$. Hadamard matrices as well as complex H-matrices have been studied by many authors. Also many methods of constructions have also been developed, yet the existence-conjectures are still open.

Williamson matrices are circulant, symmetric, commuting, $(1, -1)$-matrices A, B, C, D of order n such that $A^2 + B^2 + C^2 + D^2 = 4nI_n$. **Williamson type matrices** are their generalizations and not necessarily circulant, symmetric and commuting, rather these properties are replaced by Amicability of A, B, C, D.

Two matrices M and N are called **amicable** if $MN^T = NM^T$. Two Hadamard matrices M and N are called **special Hadamard matrices** if $MN^T + NM^T = 0$. M and N are called complex special Hadamard matrixes if $MN^* + NM^* = 0$. Special (Complex) Hadamard matrices exist for every order for which there exists a (complex) Hadamard matrix [11].

INFINITE SERIES OF BLOCK STRUCTURED COMPLEX HADAMARD MATRICES

In all of the series below 2 × 2 complex Hadamard matrices are used as blocks of the Complex Hadamard matrices.

1. Sylvester's method generalized: Well known construction of J. J. Sylvester [9] can be generalized to obtain complex Hadamard matrices of order 2^n in which each 2 × 2 block is complex Hadamard matrix by taking any 2 × 2 complex H-matrix.

2. Paley type II matrices: For any odd prime power of the form 4n+1 a block structured complex H-matrix exists and is given by H=S×A+I×B, where S is symmetric conference matrix and A & B are 2×2 special complex H-matrices.

3. Using Williamson's method:

Theorem: Existence of Williamson matrices of order n imply the existence of 2 × 2 block structured complex Hadamard matrix of order $2n$.

Proof: In [6] Singh and Topno gave a simple construction of 4 × 4 block structured Hadamard matrices contingent upon the existence of Williamson matrices. In the construction 4 × 4 blocks were given by $\hat{1},\ \hat{\underline{1}},\ \hat{\underline{2}},\ \hat{\underline{3}},\ \hat{\underline{4}}$. If we replace these by

$$a = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ b = \begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}, c = \begin{bmatrix} -i & -1 \\ -1 & -i \end{bmatrix},\ d = \begin{bmatrix} -1 & i \\ -i & 1 \end{bmatrix},\ e = \begin{bmatrix} -1 & -i \\ i & 1 \end{bmatrix}$$

respectively in the construction then we get 2 × 2 block structured Hadamard matrices of order $2n$. a, b, c, d, e have following properties:

$$ab^* + ba^* = ac^* + ca^* = ad^* + da^* = ae^* + ea^* = -2I,$$
$$bc^* + cb^* = bd^* + db^* = be^* + eb^* = dc^* + cd^* = ec^* + ce^* = de^* + ed^* = 0.$$

This construction is synonymous to the construction offered by Agaian and the resulting matrix is BCH matrix.

Cor.: Existence of Williamson matrices of order n imply the existence of half-full weighing matrices whose blocks of order 4 are also half-full weighing matrices.

Proof: By matrix representation of complex no. $a+ib=\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ for 1 and i in above construction.

4. Using Williamson type matrices

Following lemma is employed to construct the desired matrix.

Lemma: Let $a=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, b=\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, c=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, d=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Then $\begin{matrix} aa^T = cc^T = a, \; bb^T = dd^T = -b, \; ab^T = ba^T = ca^T = aa^T = 0 \\ bd^T = db^T = cd^T = dc^T = 0, \; ad^T = -cb^T = c, \; da^T = -bc^T = d \end{matrix}$

Moreover $\frac{1}{2}(1+i)(\pm a \pm b \pm c \pm d) + \frac{1}{2}(1-i)(\pm b \pm a \pm d \pm c)$ is always a 2 × 2 complex Hadamard matrix for combinations of a, b, c, d if $a \& b$ and $c \& d$ have same signs.

Proof: Straight forward verification.

Theorem: Let A, B, C, D be Williamson type matrices of order m then there exists a block structured complex Hadamard matrix of order $2m$.

Proof: Let a, b, c, d be as above lemma. Then the block structured complex Hadamard matrix can be given by

$$H = \frac{1}{2}(1+i)[A \times a + B^T \times b + C \times c + D^T \times d] + \frac{1}{2}(1-i)[A^T \times a + B \times b + C^T \times c + D \times d]$$

It can be verified easily by virtue of above lemma.

5. Using Turyn's method [10]:

First we prove the result for Real case and then generalize it for the complex case.

Lemma: If $q \equiv 1 \pmod 4$ is a prime power with $q + 1$ even, there exists Hadamard matrix of order $2(q + 1)$ which can be expressed as a block circulant matrix where blocks belong to a set of 3 skew-type H-matrices.

Proof: We follow the Turyn's method as given in Hall [4]. Williamson matrices are taken as $A = I + R$, $B = I - R$, $C = D = S$, where R & S satisfy $R^2 + S^2 = qI_{q+1}$. Then W-matrices are

$$W_1 = \frac{1}{2}(A + B + C - D) = I$$

$$W_2 = \frac{1}{2}(A + B - C + D) = I$$

Hence W_3 and W_4 matrices, for some k, must be of the form

$$W_3 = I + 2(\rho_1\omega_{i_1} + \rho_2\omega_{i_2} + \ldots + \rho_k\omega_{i_k})$$

$$W_4 = I + 2(\rho_{k+1}\omega_{i_{k+1}} + \ldots + \rho_m\omega_{i_m}) \text{ for some } k$$

where ρ_i $(1 \le i \le m) \in \{-1, +1\}$, $\omega_{i_r} = \alpha^{i_r} + \alpha^{m-i_r}$, $\alpha = \text{circ}(0\ 1\ 0\ \ldots 0)$.

Now Williamson matrices are

$$A = \frac{1}{2}(W_1 + W_2 + W_3 - W_4)$$

$$= I + \sum_{r=1}^{k} \rho_r \omega_{i_r} - \sum_{r=k+1}^{m} \rho_r \omega_{i_r}, \text{ where } m = \frac{n-1}{2}.$$

$$B = \frac{1}{2}(W_1 + W_2 - W_3 + W_4)$$

$$= I - \sum_{r=1}^{k} \rho_r \omega_{i_r} + \sum_{r=k+1}^{m} \rho_r \omega_{i_r}$$

$$C = I + \sum_{r=1}^{k} \rho_r \omega_{i_r} + \sum_{r=k+1}^{m} \rho_r \omega_{i_r} = D$$

$$\Rightarrow \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \times I + \sum_{r=1}^{k} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \rho_r \times \omega_r + \sum_{r=k+1}^{m} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rho_r \times \omega_r . \quad \text{...(1)}$$

Since $\rho_r \in \{-1, +1\}$

$$\int \textit{ i.e.,} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \rho_r \in \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rho_r \in \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

We denote [1 1 1 1], [–1 1 1 1], [1 –1 1 1] as $\overline{1}$, $\underline{1}$, $\underline{2}$, respectively. Then (1) reduces to

$$\Rightarrow \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \overline{1} \times I + \sum_{r=1}^{k} \underline{2}\, \rho_r \times \omega_r + \sum_{r=k+1}^{m} \underline{1}\, \rho_r \times \omega_r \quad \text{... (2)}$$

with $\underline{2}\rho_r \in \{+\underline{2}, -\underline{2}\}$ and $\underline{1}\rho_r \in \{+\underline{1}, -\underline{1}\}$.

Let $V = \{\overline{1}, \pm \underline{1}, \pm \underline{2},\}$. We define the product which is given by the mappings from $V \times V \rightarrow \{0, 4\}$ such that

$$\left. \begin{array}{l} \overline{1}^2 = \overline{1}\ \overline{1}^T,\ \underline{i}^2 = \underline{i}\underline{i}^T,\ i \in \{1, 2\} \\ \underline{i}\,\underline{j} = \underline{i}\underline{j}^T + \underline{j}\underline{i}^T,\ I \neq j,\ i, j \in \{1, 2\} \\ \overline{1}_i = \overline{1}_i^T + \underline{i}\,\overline{1}^T,\ i \in \{1, 2\} \end{array} \right\} \quad \text{...(3)}$$

We define a mapping $f : V \to S$, the set of skew symmetric Hadamard matrices of order 4, given by

$$F([a\ b\ c\ d]) = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} \qquad ...(4)$$

Let $f(\bar{1}) = \hat{1}$ and $f(\underline{i}) = \underline{\hat{i}}$ for $i = 1, 2$. Then it is easy to verify that

$$F(\underline{i})\, f(\underline{j}) = \underline{ij}I \quad i.e.\ \ \hat{i}\,\hat{j} = \underline{ij}I \qquad ...(5)$$

provided multiplication on the left side of (4) is defined as

$$\left.\begin{aligned} &\hat{1}^2 = \hat{1}\,\hat{1}^T,\ \hat{i}^2 = \underline{\hat{i}\ \hat{i}^T},\ i \in \{1, 2\} \\ &\hat{i}\ \hat{j} = \hat{i}\ \hat{j}^T + \hat{j}\ \hat{i}^T,\ i \neq j,\ i, j \in \{1, 2\}, \\ &\hat{1}\ \underline{\hat{i}} = \hat{1}\ \underline{\hat{i}^T} + \underline{\hat{i}}\ \hat{1}^T,\ i \in \{1, 2\} \end{aligned}\right\} \qquad ...(6)$$

i.e. $\bar{1}$ and $\underline{i}$ ($i = 1, 2$) have the same algebraic properties as those of $f(\bar{1}) = \hat{1}$ and $f(\underline{i}) = \underline{\hat{i}}$ ($i = 1, 2$).

Using (3) we can obtain $f(\bar{1}) = \hat{1}$ & $f(\underline{i}) = \underline{\hat{i}}$ ($i = 1, 2$) as

$$\hat{1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix}, \quad \underline{\hat{1}} = \begin{bmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{bmatrix}, \quad \underline{\hat{2}} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

Since $\bar{1}\,\underline{i} = 4$; $i = 1, 2\,\underline{ij} = 0$ for $i \neq j$, $i, j \in \{1, 2\}$ we have by (4): $f(\bar{1}) f(\underline{i}) = 4I$, $f(\underline{i}) f(\underline{j}) =$ O (Null matrix of order 4) and $f(\underline{i}^2) = f(\bar{1}^2) = 4I$

i.e.
$$\left.\begin{aligned} &\hat{1}\ \underline{\hat{i}} = 4I,\ i \in \{1, 2\} \\ &\underline{\hat{i}}\ \underline{\hat{j}} = \mathrm{O},\ i \neq j,\ i, j \in \{1, 2\} \\ &\hat{1}^2 = \underline{\hat{i}^2} = 4I,\ i \in \{1, 2\} \end{aligned}\right\} \qquad ...(7)$$

Now from (2) & (4)

$$f\left(\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}\right) = f(\bar{1}) \times I_n + \sum_{r=1}^{k} f(\underline{2}\rho_r) \times \omega_r + \sum_{r=k+1}^{m} f(\underline{1}\rho_r) \times \omega_r \qquad ...(8)$$

is a Hadamard matrix where $f(\bar{1})$ & each $f(\underline{i}\rho_r)$; $i = 1, 2$, are also Hadamard matrices of order 4. Since the algebra of expression on the right hand side of (8) is same as the algebra of the expression

$$I_n \times f(\bar{1}) \; + \; \sum_{r=1}^{k} \omega_r \times f(\underline{2}\rho_r) + \sum_{r=k+1}^{m} \omega_r \times f(\underline{1}\rho_r) \; = \; H_{4n} \qquad \text{...(9)}$$

H_{4n} is an H-matrix of order $4n$ whose blocks are $f\ (\bar{1}) = \hat{1}$, $f\,(\underline{i}) = \underline{\hat{i}}$; $i = 1, 2$ which are 4×4 skew-type H-matrices. Resulting matrices are BCH matrices.

Now we generalize this result for the complex case.

Theorem: When $q \equiv 1 \pmod 4$ is a prime power with $q + 1$ even, there exists Hadamard matrix of order $2\,(q + 1)$ which can be expressed as a block circulant matrix where blocks belong to a set of three 2×2 complex Hadamard matrices.

Proof: Replace $\hat{1}$, $\underline{\hat{1}}$, $\underline{\hat{2}}$, by $a = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}$, $c = \begin{bmatrix} -i & -1 \\ -1 & -i \end{bmatrix}$ respectively in the expression (9) above, we get the desired result.

Cor. : For $q \equiv 1 \pmod 4$ a prime power with $q + 1$ even, there exist half-full weighing matrix with precisely three blocks of half-full weighing matrices of order 4.

Note: Although Paley type II matrices and Williamson matrices have proven to be equivalent in the real case [5], their equivalency has not been discussed in complex case in this paper.

6. Using New Method

Theorem: Let M and N be two amicable (1, – 1)-matrices of order n, n odd such that $MM^T + NN^T = 2nI_n$ then there exists complex Hadamard matrix of order 4n whose blocks are 2×2 complex H-matrices.

Proof: Take $A = (M + N)/2$, $B = (M - N)/2$ and let C and D be special complex H-matrices of order 2 then $H_{4n} = A \times C + B \times D$ is the required H-matrix.

Cor.: If special complex H-matrices are of order t then H_{2tn} exists.

Lemma: Any Hadamard matrix of the form $\begin{bmatrix} M & N \\ -N & M \end{bmatrix}$ supplies the need of M and N in the above theorem.

NB. This result is generalization of the unpublished work of Singh and Sahay [8].

Conclusion

We have discussed some infinite series of complex Hadamard matrices in which three are new. Moreover while constructing an infinite series we have also constructed a series of block-structured Hadamard matrix. Also two infinite series of half-filled weighing matrices have emerged whose blocks are also half-full.

Future prospects

Search for block-structured Hadamard matrices can be taken up for the generalized case.

References

1. Agaian, S. S., Algorithm of orthogonal matrices fast transform, *Proc. Fifth Eur. Meeting on Cybern. and Systems Res*., Vol. **8**, Ser. **B**, 317-321 (1980).
2. Agaian. S. S., *Hadamard Matrices and their Applications*, Springer-Verlag, Berlin Heidelberg, New York Tokyo (1985).
3. Craigen, R., website http://home.cc.umanitoba.ca/~craigenr/PWP/.
4. Hall (Jr.), M., *Combinatorial Theory*, Wiley-Interscience, **2nd** edition (1988).
5. Horadam, K. J., *Hadamard matrices and their Applications*, Princeton University Press, Princeton, NJ, p.18 (2007).
6. Singh, M. K., Topno, S. N., On the construction of Hadamard Matrices of order $4n$ (n odd, $n \geq 3$) with Hadamard blocks of order 4, *Acta Ciencia Indica*, Vol. **XL** M, No. **3**, 309 (2014).
7. Singh, M.K., Topno, S.N., Paswan, T., Anticirculant structured block weighing matrices from Williamson matrices, unpublished.
8. Singh, M.K., and Sahay, G., A new conjecture related to Hadamard matrices, unpublished.
9. Sylvester, J.J., Thoughts on inverse orthogonal matrices, simultaneous sign successions, and tessellated pavements two or more colours, with applications to Newton's rule, ornamental tile-work, and the theory of numbers, *Phil. Mag*., **34 (1)**, 461-475 (1867).
10. Turyn, R. J., An infinite Class of Williamson Matrices, *Jour. of Comb. Theory*, **(A) 12**, 319-321 (1972).
11. Wallis, W. D., Street. A. P., Wallis, J. S., Combinatorics : Room Squares, Sum-Free Sets, *Hadamard Matrices*, Springer-Verlag, Berlin Heidelberg, New York, p. 296 (1972).

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