ON SOME SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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It is found that without suitable initial conditions, system of partial differential equations can't be solved by Decomposition and Variational Iteration Methods. In this paper, some problems show the non applicability of ADM & VIM in absence of initial values although they are solved by transforms.

KEYWORDS : Adomian Decomposition Method (ADM); Variational Iteration Method (VIM).

INTRODUCTION

Systems of linear [1] or nonlinear PDEs [1]–[3] have gained concern in evolution equations [4] which are used to describe propagation of waves in studying shallow water waves [5]. They are also used in investigating chemical reaction diffusion model of Brusselator [6]. The essential features of these kinds of systems have wide applications. The existing commonly used techniques such as method of characteristics & the Riemann invariants [7] encountered problems due to the required size of computation work, in particular, when these systems have many PDEs [7]. To avoid these difficulties, we apply Adomian decomposition method for studying the systems involving several PDEs. VIM proposed by J. H. He [8] in the later 1990s, is also an effective method for solving these kinds of systems of PDEs. It has successfully been applied in Engineering field, long wave [9] and chemical reaction diffusion model. Laplace transform technique is also a very useful tool for solving such kind of problems.

The aim of this study is to draw attention on some problems which are found to be unsolvable by ADM and VIM in absence of proper initial conditions although the solution can be easily obtained if Laplace transform is applied to them.

ADOMIAN DECOMPOSITION METHOD

ADM [10] is a well–renowned and systematic semi–analytic method for solving linear, nonlinear, deterministic , stochastic operator equations which include ODEs, PDEs, integral equations, delay differential equations, integro- differential equations [11] etc. It permit us to solve both initial value problems (IVP) [12] and boundary value problems (BVP) [13] without restrictive assumptions required by linearization [14] and perturbation [15] and ad–hoc assumption such as initial term guessing a set of basic function [16].

Consider the differential equation,

$$L\,u + R\,u + N\,u = g\,(t)$$

where L is the highest order linear differential operator, assumed to be invertible. R is the linear differential operator of order lesser than that of L, Nu is defined as the non linear term and g is a source term.

Applying the inverse operator L^{-1} and using given conditions, we find

$$u = \psi - L^{-1}[R\,u] - L^{-1}[Nu]$$

where the function ψ represents the terms arising from integrating the source term g using the given conditions.

ADM defines the solution u by the series

$$u = \sum_{i=0}^{\infty} u_i$$

where the components u_i are determined recursively by using the relations,

$$u_0 = \psi$$
$$u_{k+1} = -L^{-1}[R(u_k)] - L^{-1}[N(u_k)], \quad k \geq 0,$$

The non linear operator $F(u)$ may be decomposed into a sum of an infinite series of polynomials as

$$F(u) = \sum_{k=0}^{\infty} A_k$$

where A_k, which depends on u_0, u_1, u_2, u_3,..... are the Adomian polynomials [17]–[19] defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{k=0}^{\infty} u_k \lambda^k \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, 3, ...$$

VARIATIONAL ITERATION METHOD

To solve the differential equation $Lu(t) + Nu(t) = g(t)$, a correctional functional is constructed as :

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda[Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)]\,d\tau$$

where λ is Lagrangian multiplier. Optimally, λ may be found, with the help of variational theory. The subscript n shows the n^{th} order approximation, $\tilde{u}_n$ is used as a restricted variation *i.e.* $\delta \tilde{u}_n = 0$. The successive approximation u_{n+1}, $n \geq 0$, of the solution u are readily obtained upon using any selective function u_0 [20].

The exact solution is obtained as the limit of the resulting successive approximations, $u = \lim_{n\to\infty} u_n$.

Laplace transform technique

If $F(t)$ and its first $(n-1)$ derivatives are continuous functions for all $t \geq 0$ and are of exponential order b as $t \to \infty$ and if $F_n(t)$ is of class A then Laplace transform of $F_n(t)$ exists when $p > t$ given by

$$L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1} F(0) - p^{n-2} F'(0) - \ldots\ldots\ldots F^{(n-1)}(0).$$

For two independent variables, $L[u_t(x,t)] = p\,L[u(x,t)] - u(x,0)$ where L is the Laplace transform operator.

Numerical problems

(i) PDE: $\xi_x + \xi_t + \zeta_x - \zeta_t = 2(\cos x + \cos t)$, $\quad \xi_x - \xi_t + \zeta_x + \zeta_t = 2(\cos x - \cos t)$

IC : $\xi(x, 0) = \zeta(x, 0) = \sin x$.

It is observed that the above problem cannot be solved by ADM and VIM with these conditions. So applying Laplace transform on both sides, we get

$$\bar{\xi}(x,p) + \bar{\zeta}(x,p) = \frac{2}{p}\sin x \qquad \ldots (1)$$

and

$$\bar{\xi}(x,p) - \bar{\zeta}(x,p) = \frac{2}{p^2+1} \qquad \ldots (2)$$

where, $\bar{\xi} = L(\xi)$ and $\bar{\zeta} = L(\zeta)$; L is the Laplace transform operator.

Solving equations (1) and (2) and then taking inverse Laplace transform, we get,

$$\xi(x,t) = \sin x + \sin t \quad \text{and} \quad \zeta(x,t) = \sin x - \sin t$$

(ii) PDE: $\xi_x + \xi_t + \zeta_x - \zeta_t = 2(e^x + e^t)$, $\quad \xi_x - \xi_t + \zeta_x + \zeta_t = 2(e^x - e^t)$

IC: $\xi(x, 0) = e^x + 1$, $\qquad \zeta(x, 0) = e^x - 1$

Solving as in problem (i), $\quad \bar{\xi}(x,p) + \bar{\zeta}(x,p) = \frac{2}{p}e^x \qquad \ldots (3)$

and

$$\bar{\xi}(x,p) - \bar{\zeta}(x,p) = \frac{2}{p-1} \qquad \ldots (4)$$

Solving equations (3) and (4) and then taking inverse Laplace transform, we get,

$$\xi(x,t) = e^x + e^t \quad \text{and} \quad \zeta(x,t) = e^x - e^t$$

(iii) PDE: $\xi_x + \xi_t + \zeta_x - \zeta_t = 4e^x \cosh t$, $\qquad \xi_x - \xi_t + \zeta_x + \zeta_t = 0$

IC: $\xi(x, 0) = \zeta(x, 0) = e^x$

Similarly, $\quad \bar{\xi}(x,p) + \bar{\zeta}(x,p) = \frac{2p}{p^2-1}e^x \qquad \ldots (5)$

and

$$\bar{\xi}(x,p) - \bar{\zeta}(x,p) = \frac{2}{p^2-1}e^x \qquad \ldots (6)$$

Solving equations (5) and (6) we get,

$$\xi(x, t) = e^{x+t} \text{ and } \zeta(x, t) = e^{x-t}$$

(iv) PDE: $\xi_x + \zeta_x + \xi_t - \zeta_t + 2(\sin x + \sin t) = 0$, $\xi_x + \zeta_x - \xi_t + \zeta_t + 2(\sin x - \sin t) = 0$

IC: $\xi(x, 0) = \cos x + 1$, $\zeta(x, 0) = \cos x - 1$.

Here,
$$\bar{\xi}(x, p) + \bar{\zeta}(x, p) = \frac{2\cos x}{p} \quad \dots (7)$$

and
$$\bar{\xi}(x, p) - \bar{\zeta}(x, p) = \frac{2p}{p^2+1} \quad \dots (8)$$

Solving equations (7) and (8), we get,

$$\xi(x, t) = \cos x + \text{cost} \quad \text{and} \quad \zeta(x, t) = \cos x - \cos t$$

Discussion

Adomian decomposition method is more quantitative than qualitative and analytic and does not need either linearization or perturbations, discretization or consequent computer–intensive calculations. But ADM has certain flaws. A series solution, practically a truncated series solution, is obtained by using ADM. This series often coincides with the Taylor's expansion of the true solution at the point $x = 0$, in the initial value case. Although the series can be rapidly convergent in a very small region, it has very slow convergence rate in the wider region, we examine and the truncated series solution is an inaccurate solution in that region, which will greatly restrict the application area of this method. ADM is not the appropriate method in the absence of initial conditions. If the problem is a boundary value problem, we need to find the initial value first, to apply Adomian decomposition method. For the oscillatory systems, Laplace transformation of Adomian series solution has some specific properties. So we use Laplace transform to obtain the analytic solution and to improve the accuracy of ADM.

Similarly, variational iteration method is also very useful for initial value problems. For linear problems, exact solutions can be obtained by only one iteration step due to the fact that Lagrange's multiplier can easily be identified. In absence of initial values, it is imperative to obtain them for applying VIM. Transforms can easily be applied to these systems of Partial differential equations where only the boundary conditions are provided. However Laplace transform is not very useful for solving nonlinear equations. We have discussed some numerical examples above to show this inability of these so-called semi-analytic Adomian decomposition method and Variational iteration methods.

Conclusion

It is found that Adomian decomposition method and Variational iteration method were unable to solve some systems of partial differential equations with given boundary conditions. Laplace transform is applied to solve them. The results obtained are accurate and in closed form. ADM and VIM may be applied by giving initial conditions viz. $\xi(0, t)$. This finding may be used further for improving the applicability of these two prominent methods. These methods may be made more applicable for directly attacking and solving the boundary value problems without a need of initial value.

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