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SOME SPECIFIC BCH–ALGEBRAS

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Hu and Li [1] introduced the concept of a BCH–algebra in 1983 as a generalization of the concepts BCK/BCI-algebras. Here we discuss method to obtain a BCH-algebra from given two BCH-algebras. Such methods are useful in developing the theory of BCH-algebras.

Introduction

Definition (1.1) : An algebra $(X; *, 0)$ of type (2, 0) is called a BCH-algebra if the following conditions are satisfied :

(BCH 1) $x * x = 0$;

(BCH 2) $x * y = 0 = y * x \Rightarrow x = y$;

(BCH 3) $(x * y) * z = (x * z) * y$;

for all $x, y, z \in X$.

The concept of a BCH-algebra is a generalization of the concepts BCI-algebra and BCK-algebra in the following sense.

Definition (1.2) : A BCH-algebra $(X; *, 0)$ is a BCI-algebra if it also satisfies condition

(BCI 1) $((x * y) * (x * z)) * (z * y) = 0$

for all $x, y, z \in X$.

Definition (1.3) : A BCI-algebra $(X; *, 0)$ is a BCK-algebra if it satisfies the condition

(BCK 1) $0 * x = 0$ for all $x \in X$.

Definition (1.4) : A BCH-algebra $(X; *, 0)$ is said to be

(A) non-negative if $0 * x = 0$ for all $x \in X$;

(B) non-singular if $0 * x = x$ for all $x \in X$;

(C) proper if it does not satisfy condition (BCI 1) .

Some properties of a BCH-algebra are as follows:

Proposition (1.5) : Let $(X; *, 0)$ be a BCH-algebra.

Then the following hold **:**

(BCH 4) $x * 0 = x$;

(BCH 5) $x * 0 = 0$ implies $x = 0$;

(BCH 6) $0 * (x * y) = (0 * x) * (0 * y)$;

(BCH 7) $(x * (x * y)) * y = 0$;

for all $x, y, \in X$.

Definition (1.5) : A non-empty subset S of a BCH-algebra $(X; *, 0)$ is called a subalgebra if $x * y \in S$ whenever $x, y \in S$.

SOME SPECIFIC BCH – ALGEBRAS

Here we discuss some specific BCH-algebras. First of all we prove the following result:

Theorem (2.1): Let $(X, *_1, 0)$ and $(Y; *_2, 0)$ be two non-negative BCH-algebras such that $X \cap Y = \{0\}$. Let '*' be a binary operation defined on $X \cup Y$, as follows:

For $a, b \in X \cup Y$, let

$$a * b = \begin{cases} a *_1 b & \text{if } a, b \in X \\ a *_2 b & \text{if } a, b \in Y \\ a & \text{otherwise} \end{cases} \quad \ldots (2.1)$$

Then $(X \cup Y; *, 0)$ is a non-negative BCH-algebra. We denote $(X \cup Y; *, 0)$ by $X \oplus Y$.

Proof : The conditions (BCH 1) $x * x = 0$ and $0 * x = 0$ are satisfied from the given definitions.

Let $a * b = 0 = b * a$. If $a, b \in X$ or $a, b \in Y$ then $a = b$. If $a \in X$ and $b \in Y$ then $a * b = 0 \Rightarrow a = 0$ and $b * a = 0 \Rightarrow b = 0$. So condition (BCH 2) of a BCH-algebra is satisfied.

Now we prove condition (BCH 3). For $a, b \in X$ and $c \in Y$ we have

$(a * b) * c = a *_1 b$ and $(a * c) * b = a *_1 b$ which imply $(a * b) * c = (a * c) * b$.

Again for $a \in X$ and $b, c \in Y$, we have

$(a * b) * c = a * c = a$ and $(a * c) * b = a * b = a$ which imply $(a * b) * c = (a * c) * b$.

This proves that $(X \cup Y; *, 0)$ is a BCH-algebra.

Corollary (2.2) : Every non-negative BCH-algebra can be extended to another non negative BCH-algebra by adjoining only one element and defining the binary operation suitably.

Proof : Let $(X, *_1, 0)$ be a non-negative BCH-algebra and let $z \notin X$ be an object. We put $Y = \{0, z\}$. Then Y is a non-negative BCH-algebra under the binary operation given by

o	0	z
0	0	0
z	z	0

Table (2.1)

Then using theorem (2.1) we see that $X \cup Y = X \cup \{z\}$ is a non-negative BCH-algebra.

Note (2.3) : In theorem (2.1) at least one of BCH-algebra X and Y must be non-negative. For, suppose that both X and Y are not non-negative. Then there exist $x \in X$ and $y \in Y$ such that

$$0 * x = s \in X \quad \text{and} \quad 0 * y = t \in Y$$

Then $(0 * x) * y = s * y = s$,

$(0 * y) * x = t * x = t$ and $s \neq t$.

So condition (BCH 3) is not satisfied.

This can be illustrated by the following example :

Example (2.4) : Let $(X, *_1, 0)$ and $(Y; *_2, 0)$ be two BCH-algebras given by the following tables

$*_1$	0	a
0	0	a
a	a	0

Table (2.2)

$*_2$	0	b
0	0	b
b	b	0

Table (2.3)

Let $Z = X \cup Y$. Applying definition (2.1) we have the Cayley table for binary operation '*' in Z :

*	0	a	b
0	0	a	b
a	a	0	a
b	b	b	0

Table (2.4)

Now $(0 * a) * b = a * b = a$

and $(0 * b) * a = b * a = b$

imply $(0 * a) * b \neq (0 * b) * a$

So $(Z; *, 0)$ is not a BCH-algebra.

Now we prove the following result:

Theorem (2.5) : Let $(X, *_1, 0)$ be a non-negative BCH-algebra and let $(Y; *_2, 0)$ be a non-singular BCH-algebras such that $X \cap Y = \{0\}$. A binary operation * be defined on $X \cup Y$ as follows

$$a * b = \begin{cases} a *_1 b & \text{if } a, b \in X \\ a *_2 b & \text{if } a, b \in Y \\ a & \text{otherwise, provide } a \neq 0. \end{cases} \quad \ldots (2.2)$$

(If $a = 0$, we regard a as an element of the BCH-algebra in which b belongs)

Then $(X \cup Y; *, 0)$ is a BCH-algebra.

Proof : For $a \neq 0$, $b \neq 0$, $c \neq 0$ and $a, b, c \in X \cup Y$, the identify

$$(a * b) * c = (a * c) * b$$

can be proved as in theorem (2.1).

Let $a, b, c \in X \cup Y$ where $a = 0$, $b \in X$, $c \in Y$. Then

$(a * b) * c = 0 * c = c$ and $(a * c) * b = c * b = c$

$(b * c) * a = b * a = b$ and $(b * a) * c = b * c = b$

$(c * a) * b = c * b = c$ and $(c * b) * a = c * a = c$

Other conditions hold as in theorem (2.2.1).

Hence $(X \cup Y; *, 0)$ is a BCH-algebra.

Remark (2.6) : Above result is not possible if X contain an element x such that $0 * x = y \neq 0$. For in this case if $0 \neq z \in Y$ then we have

$$(0 * z) * x = z * x = z$$

and

$$(0 * x) * z = y * z = y.$$

Corollary (2.7) : If S_1 and S_2 are subspaces of X and Y respectively then $S_1 \cup S_2$ is a subspace of $X \cup Y$ in theorems (2.1) and (2.2).

An as illustration of the above result we have the following example:

Example(2.8) :- Let

$0 \equiv (0\ 0\ 0\ 0)$, $1 \equiv (0\ 1\ 0\ 0)$, $2 \equiv (1\ 0\ 0\ 0)$, $3 \equiv (1\ 1\ 0\ 0)$,

$a \equiv (0\ 0\ 0\ 1)$, $b \equiv (0\ 0\ 1\ 0)$, $c \equiv (0\ 0\ 1\ 1)$.

Let $X = \{0, 1, 2, 3\}$ and $Y = \{0, a, b, c\}$. We consider BCH-algebras $(X; *, 0)$ and $(Y; o, 0)$ where binary operations * and o are defined as follows:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Table (2.5)

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Table (2.6)

Let $Z = X \cup Y$ and let binary operations $\odot$ be defined by (2.2) . Then Cayley table for $(X \cup Y; \odot, 0)$ is given by

$\odot$	0	1	2	3	a	b	c
0	0	0	0	0	a	b	c
1	1	0	1	0	1	1	1
2	2	2	0	0	2	2	2
3	3	2	1	0	3	3	3
a	a	a	a	a	0	c	b
b	b	b	b	b	c	0	a
c	c	c	c	c	b	a	0

Table (2.7)

It is easy to verify that $(X \cup Y; \odot, 0)$ is a BCH-algebra.

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