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SOME STRUCTURAL PROPERTIES IN BCH–ALGEBRAS

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The study of BCH – algebra has been initiated by Hu and Li [1] in 1983. Here we have developed some structural properties of BCH–algebras which are helpful in determining operation table with given non-singular, non-negative and *p*-semi simple elements.

Introduction

Definition (1.1) : A system $(X; *, 0)$ consisting of a non-empty set X, a binary operation * and a fixed element 0 is called a BCH–algebra if the following conditions are satisfied :

1. (BCH 1) $x * x = 0$
2. (BCH 2) $x * y = 0 = y * x$ imply $x = y$
3. (BCH 3) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$.

Definition (1.2) : In a BCH–algebra $(X; *, 0)$ a relation $\leq$ is defined as $x \leq y$ iff $x * y = 0$. This relation is a partial order relation.

Definition (1.3) : A non –empty subset S of a BCH–algebra $(X; *, 0)$ is called a subalgebra if $x * y \in S$ whenever $x, y \in S$.

Now we mention some properties of a BCH – algebra [1,2].

Theorem (1.4) : Let $(X; *, 0)$ be a BCH–algebra then following are true

4. (BCH 4) $x * 0 = x$
5. (BCH 5) $0 * (x * y) = (0 * x) * (0 * y)$
6. (BCH 6) $x * 0 = 0$ implies $x = 0$
7. (BCH 7) $(x * (x * y)) * y = 0$

for all $x, y \in X$.

Notation (1.5) : Let $(X; *, 0)$ be a BCH–algebra. Let

$$N(X) = \{x \in X : 0 * x = x\} \quad \dots (1.1)$$

$$B(X) = \{x \in X : 0 * x = 0\} \quad \dots (1.2)$$

$$P(X) = \{x \in X : 0 * (0 * x) = x\}, \qquad \ldots (1.3)$$

Then $N(X)$, $B(X)$ and $P(X)$ are called non-singular part of X, non-negative part of X and p-semi simple part of X respectively. Further $N(X) \subseteq P(X)$. Let $Q(X) = P(X) - N(X)$.

Definition (1.6) : A BCH–algebra $(X; *, 0)$ is called non – singular, non-negative and p-semi simple according as $N(X) = X$, $B(X) = X$ and $P(X) = X$.

Example (1.7) : Let $X = \{0, a\}$ and let a binary operation '*' be defined as follows:

*	0	a
0	0	a
a	a	0

Then $(X; *, 0)$ is a non-singular BCH–algebra.

PROPERTIES OF $N(X)$, $B(X)$, $P(X)$ AND $Q(X)$

First of all we see that

Lemma (2.1) : $N(X)$, $B(X)$ and $P(X)$ are BCH–subalgebras .

Proof - Let $x, y \in N(X)$. Then $0 * x = x$, $0 * y = y$. Now

$$0 * (x * y) = (0 * x) * (0 * y) \qquad \text{(by (BCH 5))}$$
$$= x * y \text{ imply } N(X) \text{ is a subalgebra.}$$

Again $x, y \in B(X) \Rightarrow 0 * x = 0$ and $0 * y = 0$.

So $0 * (x * y) = (0 * x) * (0 * y) = 0 \Rightarrow x * y \in B(X)$.

Also $x, y \in P(X) \Rightarrow 0 * (0 * x) = x$ and $0 * (0 * y) = y$.

So
$$0 * (0 * (x * y)) = 0 * ((0 * x) * (0 * y))$$
$$= (0 * (0 * x)) * (0 * (0 * y))$$
$$= x * y \Rightarrow x * y \in P(X).$$

Hence the result.

Lemma (2.2) : $x, y \in N(X) \Rightarrow x * y = y * x$.

Proof – We have $x * y = (0 * x) * y = (0 * y) * x = y * x$.

Proposition (2.3) : If $x, y \in N(X)$ and $x \neq y \neq 0$ then $x * y \neq 0$, $x * y \neq x$ and $x * y \neq y$.

Proof – If possible, suppose $x * y = 0$. Then $y * x = x * y = 0$. So

(BCH 2) imply $x = y$ which is a contradiction. So $x * y \neq 0$.

Let $x * y = x$. Then $(x * y) * x = x * x$. This gives $(x * x) * y = 0 \Rightarrow 0 * y = 0 \Rightarrow y = 0$ which is a contradiction. So $x * y \neq x$.

As above $x * y = y \Rightarrow x = 0$

which is a contradiction. So $x * y \neq y$.

Hence the result.

Corollary (2.4) : A set $X = \{0, x, y\}$ under a binary operation '*' for which $0 * x = x$ and $0 * y = y$ cannot be a BCH–algebra.

Proposition (2.5) : Let $N(X)$ be the non-singular part of a BCH–algebra $(X; *, 0)$. Let a, b, c be three non-identical and non-zero elements of $N(X)$ such that $a * b = c$. Then $a * c = b$ and $b * c = a$.

Proof : Using lemmas (2.1) and (2.2) , we have

$$a * c = c * a = (a * b) * a = (a * a) * b = 0 * b = b$$

and $$b * c = c * b = (a * b) * b = (b * a) * b = (b * b) * a$$

$$= 0 * a = a.$$

Hence the result.

Corollary (2.6) : Let $(X; *, 0)$ where $X = \{0, a, b, c\}$ be a non-singular BCH-algebra. Then the Cayley table is as follows:

*	0	*a*	*b*	*c*
0	0	*a*	*b*	*c*
a	*a*	0	*c*	*b*
b	*b*	*c*	0	*a*
c	*c*	*b*	*a*	0

This table is also unique.

Theorem (2.7) : In the binary operation table of a finite non-singular BCH-algebra no two elements of a particular row (or a particular column) are identical.

Proof : Let $0 = x_0, x_1, x_2, ..., x_{n-1}$ be n distinct elements of a non-singular BCH – algebra $(X; *, 0)$. If possible, let $x_i * x_j = x_i * x_k$ where $j \neq k$. Then

$$(x_i * x_j) * x_i = (x_i * x_k) * x_i.$$

This gives $$(x_i * x_i) * x_j = (x_i * x_i) * x_k$$

i.e., $x_j = x_k$, which is a contradiction.

Hence $$x_i * x_j \neq x_i * x_k.$$

Again $$x_i * x_1 = x_j * x_1 \quad (i \neq j)$$

$$\Rightarrow (x_i * x_1) * x_1 = (x_j * x_1) * x_1$$

$$\Rightarrow (x_1 * x_i) * x_1 = (x_1 * x_j) * x_1$$

$$\Rightarrow (x_1 * x_1) * x_i = (x_1 * x_1) * x_j$$

$$\Rightarrow x_i = x_j \text{ which is a contradiction .}$$

Hence the result.

Corollary (2.8) : If $x, y \in N(X)$ and $x \neq y \neq z$ then $x * y \neq x * z$.

Theorem (2.9): Let $(X; *, 0)$ be a BCH–algebra. Let $0 \neq a \in N(X)$ and $0 \neq b \in B(X)$. Then

(i) $a * b = a$

and (ii) either $b * a = a$ or $(b * a) * a \in B(X)$.

Proof – (i) Using (BCH 5) we have

$$0 * (a * b) = (0 * a) * (0 * b) = a * 0 = a \text{ (by (BCH 4)).}$$

Let $a * b = c$. Then

$$a * c = (0 * a) * c = (0 * c) * a = a * a = 0$$

and $$c * a = (a * b) * a = (a * a) * b = 0 * b = 0.$$

So using (BCH 2) we have $c = a$, *i.e.*, $a * b = a$.

(ii) Again $0 * (b * a) = (0 * b) * (0 * a) = 0 * a = a$.

Let $b * a = d$. We have

$$0 * (d * a) = (0 * d) * (0 * a) = a * a = 0.$$

This gives either $d * a = 0$ or $d * a \in B(X)$.

Also $a * d = (0 * a) * d = (0 * d) * a = a * a = 0$.

Thus if $d * a = 0$ then (BCH 2) gives $d = a$.

So either $b * a = a$ or $(b * a) * a \in B(X)$.

Theorem (2.10) : Let $(X; *, 0)$ be a BCH–algebra. Let

$0 \neq a \in N(X)$, $0 \neq b \in B(X)$, $c \in Q(X)$ and $0 * c = d$. Then

(i) $0 * d = c$ and $d \in Q(X)$,

(ii) $c * b = c$,

(iii) $d * (b * c) = 0$ and $c * (b * d) = 0$,

(iv) $a * c \notin N(X)$, $a * c \notin B(X)$,

(v) $b * c \notin N(X) \cup B(X)$ and $b * c \neq c$,

(vi) $c * d \neq 0$, $c * d \neq b$ and $c * d \neq c$.

Proof - (i) We have

$$0 * d = 0 * (0 * c) = c$$

and $$0 * (0 * d) = 0 * c = d. \text{ So } d \in Q(X).$$

(ii) Let $c * b = l$. Then

$$l * c = (c * b) * c = (c * c) * b = 0 * b = 0.$$

Also
$$\begin{aligned} c * l &= (0 * d) * l \quad \text{(by (i))} \\ &= (0 * l) * d \\ &= (0 * (c * b)) * d \\ &= ((0 * c) * (0 * b)) * d \\ &= (d * 0) * d = (d * d) * 0 = 0. \end{aligned}$$

So (BCH 2) implies $l = c$, *i.e.*, $c * b = c$.

(iii) Let $b * c = m$. Then

$$\begin{aligned} d * m &= (0 * c) * m \\ &= (0 * m) * c \\ &= (0 * (b * c)) * c \\ &= ((0 * b) * (0 * c)) * c \\ &= (0 * d) * c \\ &= c * c = 0. \end{aligned}$$

So $d * (b * c) = 0$.

Interchanging d and c we get $c * (b * d) = 0$.

(iv) We have

$$a * c = 0 \Rightarrow (a * c) * a = 0 * a \Rightarrow (a * a) * c = a$$
$$\Rightarrow 0 * c = a \Rightarrow d = a.$$

which is a contradiction.

Again $$a * c = a \Rightarrow (a * c) * a = a * a \Rightarrow (a * a) * c = 0$$
$$\Rightarrow (0 * c) = 0 \Rightarrow d = 0.$$

which is a contradiction.

Also $a * c = a^1 \in N(X) \Rightarrow (a * c) * a^1 = a^1 * a^1 = 0$
$$\Rightarrow (a * a^1) * c = 0.$$

Since $(a * a^1) \in N(X)$ above argument gives a contradiction.

Further, $a * c = b \in B(\mathrm{X}) \Rightarrow 0 * (a * c) = 0 * b$.
$$\Rightarrow (0 * a) * (0 * c) = 0$$
$$\Rightarrow a * d = 0$$
$$\Rightarrow (a * d) * a = 0 * a$$
$$\Rightarrow (a * a) * d = a$$
$$\Rightarrow 0 * d = a$$
$$\Rightarrow c = a.$$

which is a contradiction.

Hence the result.

We have

$$b * c = a \in N(X)$$
$$\Rightarrow 0 * (b * c) = 0 * a.$$
$$\Rightarrow (0 * b) * (0 * c) = a$$
$$\Rightarrow 0 * (0 * c) = a$$
$$\Rightarrow c = a.$$

which is a contradiction.

Again $b * c = b \in B(X)$
$$\Rightarrow 0 * (b * c) = 0 * b = 0$$
$$\Rightarrow (0 * b) * (0 * c) = 0$$
$$\Rightarrow 0 * (0 * c) = 0$$
$$\Rightarrow c = 0.$$

which is a contradiction.

Also $$b * c = c$$
$$\Rightarrow 0 * (b * c) = 0 * c = d$$
$$\Rightarrow (0 * b) * (0 * c) = d$$

$$\Rightarrow\ 0 * (0 * c) = d$$
$$\Rightarrow\ c = d.$$

which is a contradiction.

This proves the result.

We see that

$$c * d = 0 \Rightarrow (c * d) * c = 0 * c$$
$$\Rightarrow (c * c) * d = d$$
$$\Rightarrow 0 * d = d$$
$$\Rightarrow c = d.$$

which is a contradiction. So $c * d \neq 0$.

Again $\quad c * d = \text{b} \Rightarrow (c * d) * b = 0 \Rightarrow (c * b) * d = 0$

$\Rightarrow c * d = 0$ (by (ii)) which is a contradiction.

Also $\quad c * d = c \Rightarrow (c * d) * c = 0 \Rightarrow 0 * d = 0 \Rightarrow c = 0$

which is a contradiction.

Hence the result.

Corollary (2.11) : In a finite BCH–algebra X, $Q(X)$ contains even number of elements.

Example (2.12) : Let $X = \{0, a, b, c, d\}$ and let '*' be a binary operation on X such that $N(X) = \{0, a\}$, $B(X) = \{0, b\}$ and $Q(X) = \{c, d\}$. Under these conditions we wish to construct BCH – algebras. Using theorems (2.9) and (2.10) we have the following table for the binary operation :

*	0	a	b	c	d
0	0	a	0	d	c
a	a	0	a	l	m
b	b	n	0	x	y
c	c	u	c	0	v
d	d	w	d	t	0

where we have to determine l, m, n, x, y, u, v, w and t so that above table becomes a BCH – algebra.

In view of theorem (2.10) (iv) we see that

$$a * c = c \text{ or } d$$

Now $\quad l = a * c = c \Rightarrow (a * c) * a = c * a$

$$\Rightarrow (a * a) * c = c * a$$
$$\Rightarrow 0 * c = c * a \Rightarrow d = c * a,$$

i.e., $u = d$

Also $\quad a * c = c \Rightarrow 0 * (a * c) = 0 * c$

$$\Rightarrow (0 * a) * (0 * c) = d$$
$$\Rightarrow a * d = d, \text{ i.e., } m = d$$

Further, $\quad d * a = (0 * c) * a = (0 * a) * c = a * c = c$

Thus, $l = c \Rightarrow m = d$, u = d and w = c.

Similar arguments gives that if $l = a * c = d$ then $m = c$, $w = d$ and $u = c$.

In view of theorem (2.9) either $b * a = a$ or $(b * a) * a \in B(X)$. For any other value of $b * a$, $(b * a) * a \notin B(X)$. So $b * a = a$, *i.e.*, $n = a$.

Again in view of theorem (2.10) (v) we see that

$$x = b * c = d \text{ and } y = b * d = c$$

In view of theorem (2.10) (vi) we see that $c * d = a$ or d.

Let $c * d = a$. Then $(c * d) * c = a * c$

$$\Rightarrow (c * c) * d = a * c$$

$$\Rightarrow 0 * d = a * c$$

$$\Rightarrow c = a * c.$$

So $c * d = a$ is possible only when $a * c = c$.

Also in this case

$$d * c = (0 * c) * (0 * d) = 0 * (c * d)$$
$$= 0 * a$$
$$= a.$$

In other case we take $c * d = d$

In this case $(0 * c) * (0 * d) * 0 * d$

i.e., $$d * c = c$$

Thus $v = a \Rightarrow t = a$ and $v = d \Rightarrow t = c$.

Thus possible BCH–algebras with required conditions are as follows :

*	0	a	b	c	d
0	0	a	0	d	c
a	a	0	a	c	d
b	b	a	0	d	c
c	c	d	c	0	a
d	d	c	d	a	0

*	0	a	b	c	d
0	0	a	0	d	c
a	a	0	a	d	c
b	b	a	0	d	c
c	c	c	c	0	d
d	d	d	d	c	0

Theorem (2.11) : In a finite BCH-algebra X, $N(X)$ contains 2^n distinct elements, n being a natural number.

Proof : We have seen in example (1.7) that a set X containing 2 elements can be a non-singular BCH-algebra under suitable binary operation. We have also seen in corollary (2.4) that a BCH-algebra having three elements cannot be non-singular. Further, in corollary (2.6) we have seen that a set having $4 = 2^2$ elements is a non- singular BCH-algebra under a suitable binary operation.

Let $X = \{o, a, b, c\}$ be a BCH-algebra under a binary operation '*'. Let $Y = X \cup \{d\}$. In order that Y is a non singular BCH- algebra under an extended binary operation 'o' we must have $0 \, o \, d = d$. In view of theorem (2.7) $a \, o \, d$, $b \, o \, d$, $c \, o \, d$ must be distinct elements of Y. We may assume $a \, o \, d = e$, $b \, o \, d = f$ and $c \, o \, d = g$.

So the number of elements in Y is 2^3. The above arguments suggested that if $(X, *, 0)$ is a non-singular BCH-algebra containing 2^n elements, then a non-singular BCH-algebra containing X must contain $2^n \cdot 2 = 2^{n+1}$ elements. Hence the result.

References

1. Hu, Q.P. and Li, X., On BCH-algebras, *Math Seminar Notes*, **11**, 313-320 (1983).
2. Hu, Q.P. and Li, X., On proper BCH-algebras, *Math Japonica*, **30**, 659-661 (1985).
3. Ilyas, S. and Prasad, R. L., Ideals in semi commutative BCI–algebra, *Aryabhata Research J. of Phy. Sci.*, Vol. **17**, pp. 9–11 (2014).
4. Kumar, Pawan, Khan, M.J.I. and Prasad, R. L., Non-singular BCI–algebras, *Anusandhan*, Vol. **XVIII**, No. **46**, 1-5 June (2016).

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