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SEMI-COMMUTATOR GRAPHS IN BCI–ALGEBRAS

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Zahiri and Borzooei [8] have developed the concept of graph in BCI–algebras in 2012. Some other authors have also developed the concept in different methods. Here we develop the concept of graph for semi-commutator elements and present an example which shows that all the concepts are different.

KEYWORDS : BCI–algebra , graph.

Introduction

Definition (1.1) : A system $(X;\ *,\ 0)$ consisting of a non-empty set X, a binary operation $*$ and a fixed element 0 , is called a BCI – algebra if the following conditions are satisfied :

1. (BCI 1) $((x * y) * (x * z)) * (z * y) = 0$
2. (BCI 2) $(x * (x * y)) * y = 0$
3. (BCI 3) $x * x = 0$
4. (BCI 4) $x * y = 0 = y * x \Rightarrow x = y$

for all $x, y, z \in X$.

Definition (1.2) : In a BCI–algebra $(X;\ *,\ 0)$ a partial order relation $\leq$ is defined as $x \leq y$ iff $x * y = 0$.

Notation (1.3) : For a BCI–algebra $(X;\ *,\ 0)$. Let $B(X)$, $P(X)$ and $N(X)$ denote the sets

$$B(X) = \{x \in X : 0 * x = 0\} \qquad \dots (1.1)$$

$$P(X) = \{x \in X : 0 * (0 * x) = x\} \qquad \dots (1.2)$$

$$N(X) = \{x \in X : 0 * x = x\} \qquad \dots (1.3)$$

Here $B(X)$, $P(X)$ and $N(X)$ are called BCK–part of X, p-semi simple part of X and non-singular part of X respectively.

Notation (1.4): For any $a \in P(X)$, let $V(a)$ be the set

$$V(a) = \{x \in X : a * x = 0\} \qquad \dots (1.4)$$

and is called the branch of X initiated by a.

Definition (1.5) : A subset A of a BCI–algebra $(X;\ *,\ 0)$ is called semi commutator if $x * y = y * x$ for all $x, y \in A$.

Lemma (1.6) : Let $(X; *, 0)$ be a BCI–algebra. Then the following results hold;

(P 1) $x * 0 = x$;

(P 2) $x * (x * (x * y)) = x * y$;

(P 3) $(x * y) * z = (x * z) * y$;

(P 4) $0 * (x * y) = (0 * x) * (0 * y)$;

(P 5) $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$.

We also have

Theorem (1.7) : Let $(X; *, 0)$ be a BCI–algebra and let $N(X)$ be the non- singular part of X. Then

(i) $N(X)$ is a subalgebra of X,

(ii) $N(X)$ is semi-commutative, *i.e.*,

$$x, y \in N(X) \Rightarrow x * y = y * x;$$

(iii) if $x, y \in N(X)$, $x \neq y \neq 0$ then neither $x * y = 0$ nor $x * y = x$ nor $x * y = y$;

(iv) $a, b \in N(X)$ equation $a * x = b$ has a unique solution $x = a * b$ in $N(X)$.

The above results follow from results appearing in [3] and [4].

Notation (1.8) : Let X be a BCI–algebra. For $A \subseteq X$ let $U(A)$ and $L(A)$ be defined as

$$U(A) = \{x \in X : a * x = 0 \text{ for all } a \in A\},$$

$$L(A) = \{x \in X : x * a = 0 \text{ for all } a \in A\}.$$

Definition (1.9) : Let $x \in X$. Then there exists $a \in X$ such that $x \in V(a)$. Let

$$Z_x = \{y \in X : L(\{x, y\}) = \{a\}\}.$$

The set Z_x is called the a - divisor of x.

Zahiri and Borzooei [8] have established the following results.

Lemma (1.10):- Let a, b $\in$ P(X) and x, y $\in$ X.

Then (a)

$$L(\{x, a\}) = \begin{cases} a & \text{if } x \in V(a) \\ \phi & \text{otherwise} \end{cases}$$

(b) if $a \neq b$, $x \in V(a)$ and $y \in V(b)$ then $L(\{x, y\}) = \phi$

(c) $x * y = 0 \Rightarrow L(\{x\}) \subseteq L(\{y\})$ and $Z_y \subseteq Z_x$.

Definition (1.11) : Let $Y \subset X$ let and $G(Y)$ be a simple graph, whose vertices are just the elements of Y and for distinct $x, y \in Y$ there is an edge connecting x and y, denoted as xy iff $L(\{x, y\}) = \{a\}$ for some $a \in P(x)$. If $Y = X$ then $G(X)$ is called a BCI graph of X.

We also call such a graph as graph of type I.

The following definition of a graph appears in [5].

Definition (1.12) : Let $(X; *, 0)$ be a BCI – algebra with $|X| \geq 4$ and let $N(X)$ be the non-singular part of X. Let $G(N(X) - \{0\})$ be a simple graph whose vertices are non – zero element of $N(X)$ such that for non-zero district $a, b, c \in N(X)$ there are edges connecting a and b, b and c, c and a, denoted as ab, bc, ca respectively, iff $a * b = c$, $b * c = a$ and $c * a = b$. This graph is called non-singular graph of X or graph of type II.

SEMI–COMMUTATOR GRAPH

We define a graph in a BCI- algebra as follows:

Definition (2.1) : Let $(X; *, 0)$ be a BCI–algebra and let A be semi-Commutator subset of X. Then $G(A)$ is a simple graph whose vertices are in A and there are edge connecting a and b $(a \neq b)$ iff $a * b = b * a$. We denote such edge as ab. Also such graph is called semi-commutator graph of X or graph of type III.

Now we give an example in which all three types of distinct graphs can be drawn.

Example (2.2) : Let $X = Y = \{0, 1\}$ we consider binary operations $*$ and o given by

*	0	1
0	0	0
1	1	0

and

o	0	1
0	0	1
1	1	0

respectively.

Then $(X; *, 0)$ is a BCK–algebra and $(Y; \text{o}, 0)$ is a BCI–algebra. Let $Z = X \times X \times Y \times Y$.

Then Z is a BCI–algebra in which binary operation $\odot$ is extended by $*$ and o respectively for first two coordinates and last two coordinates.

Let $0 = (0\ 0\ 0\ 0)$, $a = (0\ 0\ 0\ 1)$,

$b = (0\ 0\ 1\ 0)$, $c = (0\ 0\ 1\ 1)$, $d = (0\ 1\ 0\ 0)$,

$e = (0\ 1\ 0\ 1)$, $f = (0\ 1\ 1\ 0)$, $g = (0\ 1\ 1\ 1)$,

$h = (1\ 0\ 0\ 0)$, $i = (1\ 0\ 0\ 1)$, $j = (1\ 0\ 1\ 0)$,

$k = (1\ 0\ 1\ 1)$, $l = (1\ 1\ 0\ 0)$, $m = (1\ 1\ 0\ 1)$,

$n = (1\ 1\ 1\ 0)$, $p = (1\ 1\ 1\ 1)$.

The binary operation $\odot$ in Z is presented by the following table.

	$\odot$	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p		
	0	0	a	b	c	0	a	b	c	0	a	b	c	0	a	b	c		
	a	a	0	c	b	a	0	c	b	a	0	c	b	a	0	c	b		
	b	b	c	0	a	b	c	0	a	b	c	0	a	b	c	0	a c	c	b
a	0	c	b	a	0	c	b	a	0	c	b	a	0	d	d	e	f	g	0
a	b	c	d	e	f	g	0	a	b	c									
	e	e	d	g	f	a	0	c	b	e	d	g	f	a	0	c	b		
	f	f	g	d	e	b	c	0	a	f	g	d	e	b	c	0	a		
	g	g	f	e	d	c	b	a	0	g	f	e	d	c	b	a	0		
	h	h	i	j	k	h	i	j	k	0	a	b	c	0	a	b	c		
	i	i	h	k	j	i	h	j	k	a	0	c	b	a	0	c	b		
	j	j	k	h	i	j	k	h	i	b	c	0	a	b	c	0	a		
	k	k	j	i	h	k	j	i	h	c	b	a	0	c	b	a	0		
	l	l	m	n	p	h	i	j	k	d	e	f	g	0	a	b	c		
	m	m	l	p	n	i	h	k	j	e	d	g	f	a	0	c	b		

n	n	p	l	m	j	k	h	i	f	g	d	e	b	c	0	a
p	p	n	m	l	k	j	i	h	g	f	e	d	c	b	a	0

Here

$$B(Z) = \{0, \ d, \ h, \ l\}$$
$$P(Z) = \{0, \ a, \ b, \ c\} = N(Z)$$

$$V(0) = \{0, \ d, \ h, \ l\}$$
$$V(a) = \{a, \ e, \ i, \ m\}$$
$$V(b) = \{b, \ f, \ j, \ n\}$$
$$V(c) = \{c, \ g, \ k, \ p\}.$$

We have

$$L(\{0, \ d\}) = L(\{0, \ h\}) = L(\{0, \ l\})$$
$$= L(\{d, \ h\}) = L(\{d, \ l\})$$
$$= L(\{h, \ l\}) = \{0\}.$$

So there exist edges 0d, 0h, 0l, dh, dl, hl.

Similarly

$$L(\{a, \ e\}) = L(\{a, \ i\}) = L(\{a, \ m\}) = L = (\{e, \ i\})$$
$$= L(\{e, m\}) = L(\{i, \ m\}) = \{a\}.$$

So there exist edges ae, ai, am, ei, em, im.

Also $L(\{b, \ f\}) = L(\{b, \ j\}) = L(\{b, \ n\}) = L = (\{f, \ j\})$

$$= L(\{f, \ n\}) = L(\{j, \ n\}) = \{b\}.$$

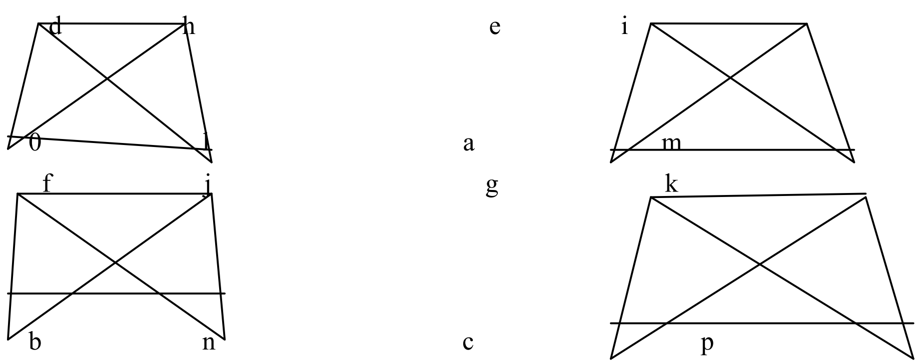
So there exist edges bf, bj, bn, fj, fn, jn.

Also $L(\{c, \ g\}) = L(\{c, \ k\}) = L(\{c, \ p\}) = L = (\{g, \ k\})$

$$= L(\{g, p\}) = L(\{k, \ p\}) = \{c\}.$$

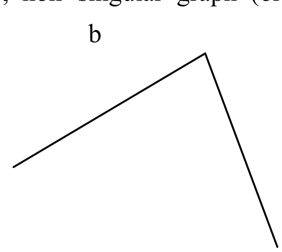
So there exist edges cg, ck, cp, gk, gp and kp.

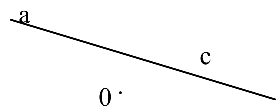
Thus the BCI graph (or graph of type I) is given by



Non zero elements of N(Z) are a, b, c. Also $a \odot b = c$, $b \odot c = a$ and $c \odot a = b$.

So, non -singular graph (or graph of type II) is given by



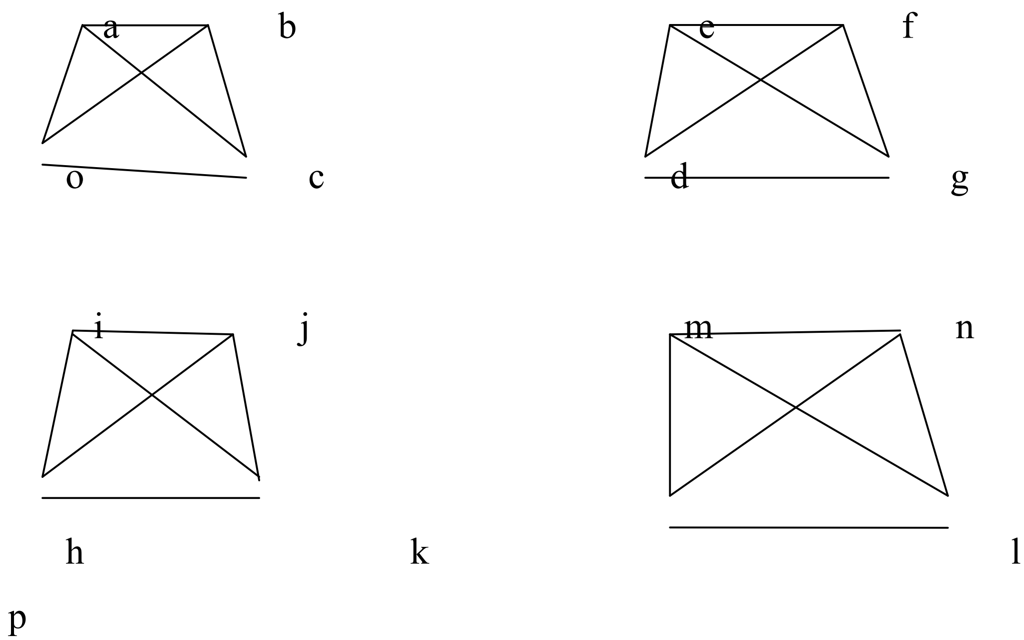


Further, semi- commutator subsets of Z are

A = {0, a , b, c}, B = {d, e, f, g}

C = {h, i, j, k}, D = {l, m, n, p}

Thus semi commutator graphs (or graph of type III) of Z are represented as



NOTE(2.3):- graphs of type I, II and III are different and the graphs are simple

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