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ON CERTAIN SPECIAL VECTOR FIELDS IN A FINSLER SPACE

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Concurrent vector fields in a Finsler space were defined and studied in (1950) by Tachibana [5]. In (1974), Matsumoto and Eguchi [1] continued this study further. Rastogiand Dwivedi [3] in (2004) studied the existence of concurrent vector fields in a Finsler space and found that the definition given earlier is untenable. This led to an alternative definition of concurrent vector fields [3] in a Finsler space. The purpose of the present paper is to define certain special vector fieldsin a Finsler space and study some of their properties vis-à-vis concurrent vector fields.

Introduction

Let F^n be an n-dimensional Finsler space with metric function $L(x, y)$, metric tensor $g_{ij}(x, y)$, angular metric tensor $h_{ij} = g_{ij} - l_i l_j$, where $l_i = \Delta_i L$ and torsion tensor $C_{ijk} = (1/2)\Delta_k g_{ij}$. The h- and v-covariant derivatives of a tensor field V^i_j are respectively given by Rund [4] as

$$V^i_{j/k} = \delta_k V^i_j + V^m_j F^i_{mk} - V^i_m F^m_{jk} \qquad \ldots (1.1)$$

and

$$V^i_{j//k} = \Delta_k V^i_j + V^m_j C^i_{mk} - V^i_m C^m_{jk} \qquad \ldots (1.2)$$

where $\delta_k = \partial_k - N^m_k \Delta_m$, ∂_j and Δ_j respectively denote partial differentiation with respect to x^j and y^j.

The torsion tensors A_{ijk} and P_{ijk} are given as $A_{ijk} = LC_{ijk}$, $P_{ijk} = A_{ijk/r} l^r = A_{ijk/0}$, $l^i = L^{-1} y^i$. In $F^3 C_{ijk}$ is expressed as Matsumoto [2]:

$$C_{ijk} = C_{(1)} m_i m_j m_k - C_{(2)}(m_i m_j n_k + m_j m_k n_i + m_k m_i n_j)$$
$$+ C_{(3)}(m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) + C_{(2)} n_i n_j n_k \qquad \ldots (1.3)$$

Now we shall give the definition of concurrent vector fields [3]:

Definition 1. A vector field $X^i(x)$ in a Finsler space F^n is called concurrent vector field if it satisfies $X^i_{/k} = -\delta^i_k$ and $X^i A_{ijk} = \alpha h_{jk}$, where α is a scalar function of x and y and other terms have their usual meaning.

Two dimensional Finsler space

It is known that in a two dimensional Finsler space F^2, $g_{ij} = l_i l_j + m_i m_j$, $h_{ij} = m_i m_j$, $l_{i/j} = 0$, $m_{i/j} = 0$, $l_{i//j} = L^{-1} m_i m_j$ and $m_{i//j} = -L^{-1} l_i m_j$. Let $X^i(x)$ be a vector field in F^2, which is a function of x alone, then we shall give following definition**:**

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Def. (2.1). A vector field X^i (x) in F^2, shall be called a special vector field of first kind, if it satisfies $X^i_{/j} = -\delta^i_j$ and

$$X^i h_{ij} = \Theta_j \qquad \dots (2.1)$$

where Θ_j is a non-zero vector field in F^2.

If we assume

$$X^i = A\, l^i + B\, m^i \qquad \dots (2.2)$$

where A and B are scalars, then we can observe

$$X^i l_i = A,\; X^i\, m_i = B \qquad \dots (2.3)$$

Substituting the value of h_{ij} in (2.1) and using equation (2.3), we can obtain

$$B\, m_j = \Theta_j \qquad \dots (2.4)$$

From equation (2.3), we can observe that $A_{/j} = -\, l_j$ and $B_{/j} = -\, m_j$, which leads to $X^i\, A_{/i} = -\, A$ and $X^i\, B_{/i} = -\, B$. Also equation (2.4) can alternatively be expressed as $B\, B_{/j} = -\, \Theta_j$ or $B^2{}_{/j} = -\, 2\Theta_j$. By taking h-covariant differentiation of equation (2.1), we can easily obtain

$$\Theta_{j/k} = -\, h_{kj}, \qquad \dots (2.5)$$

which shows that $\Theta_{j/k}$ is symmetric in j and k. Also it is easy to observe

$$\Theta_{j/k} l^j = 0,\; \Theta_{j/k} l^k = 0,\; \Theta_{j/k} m^j = -\, m_k,\; \Theta_{j/k} m^k = -\, m_j \text{ and } \Theta_{j/k/h} = 0.$$

By taking v-covariant derivative of (2.3), we get $A_{//j} = L^{-1}\, B\, m_j$ and $B_{//j} = (BC - L^{-1}A)\, m_j$, showing that $A_{//j} l^j = 0$, $B_{//j} l^j = 0$, $A_{//j} m^j = B\, L^{-1}$ and $B_{//j} m^j = B\, C - L^{-1}A$. Taking v-covariant derivative of (2.1) we get

$$\Theta_{j//k} = (B\, C - L^{-1}A)\, m_j m_k - L^{-1}\, B\, l_j m_k\,, \qquad \dots (2.6)$$

which gives $\Theta_{j//k} l^j = -\, L^{-1}\, B\, m_k$, $\Theta_{j//k} l^k = 0$, $\Theta_{j//k} m^j = (B\, C - L^{-1}A)\, m_k$ and $\Theta_{j//k} m^k = (B\, C - L^{-1}A)\, m_j - L^{-1}\, B\, l_j$. Further from equation (2.6) we can obtain

$$\Theta_{j//k} - \Theta_{k//j} = L^{-1}B\, (l_k m_j - \; l_j m_k) \qquad \dots (2.7)$$

Hence we have:

Theorem (2.1). In a 2-dimensional Finsler space F^2, a special vector field of first kind, $X^i(x)$ is such that $\Theta_{j/k}$ is symmetric in j and k, while $\Theta_{j//k}$ is non- symmetric in j and k and satisfies (2.7).

Now $X^i C_{ijk} = X^i\, C\, m_i m_j m_k = X^i C\, h_{ij} m_k$, therefore by virtue of (2.1), we get

$$X^i A_{ijk} = LBC\, h_{jk} \qquad \dots (2.8)$$

Comparing equation (2.8) with definition 1, we can observe that $\alpha = LBC$. Hence we have:

Theorem 2.2. In a two dimensional Finsler space F^2, a special vector field of first kind is also a concurrent vector field, whose coefficient is given by $\alpha = LBC$.

THREE DIMENSIONAL FINSLER SPACE

In a three dimensional Finsler space F^3, following Matsumoto [2], we have $g_{ij} = l_i l_j + m_i m_j + n_i n_j$, $h_{ij} = m_i m_j + n_i n_j$, $l_{i/j} = 0$, $m_{i/j} = n_i h_j$, $n_{i/j} = -m_i h_j$, $l_{i//j} = L^{-1} h_{ij}$, $m_{i//j} = L^{-1}(-l_i m_j + n_i v_j)$ and $n_{i//j} = -L^{-1}(l_i n_j + m_i v_j)$. Let $X^i(x)$ be a vector field in F^3, which is a function of x alone, then we give the following definition:

Def. (3.1). A vector field $X^i(x)$ in F^3, shall be called a special vector field of first kind, if it satisfies $X^i_{/j} = -\delta^i_j$ and

$$X^i h_{ij} = \varphi_j, \qquad \ldots (3.1)$$

where φ_j is a vector field in F^3.

If we assume

$$X^i = A\, l^i + B\, m^i + D n^i, \qquad \ldots (3.2)$$

where A, B and D are scalars, we can observe

$$X^i\, l_i = A,\ X^i\, m_i = B,\ X^i n_i = D \qquad \ldots (3.3)$$

Substituting the value of h_{ij} in equation (3.1) and using (3.3), we can obtain

$$B\, m_j + D\, n_j = \varphi_j, \qquad \ldots (3.4)$$

From equations (3.2) and (3.3), we can observe that $A_{/j} = -l_j$, $B_{/j} = D\, h_j - m_j$ and $D_{/j} = -(B\, h_j + n_j)$, which lead to $X^i A_{/I} = -A$, $X^i B_{/I} = D\, h_i\, X^i - B$ and $X^i D_{/I} = -(B\, h_i\, X^i + D)$. From these results and equation (3.4), we can easily obtain

$$(B^2 + D^2)_{/j} + 2\varphi_j = 0, \qquad \ldots (3.5)$$

and

$$(B^2 + D^2)_{/0} = 0,\ (B^2 + D^2)_{/j} m^j + 2B = 0,\ (B^2 + D^2)_{/j} n^j + 2D = 0. \qquad \ldots (3.6)$$

Hence we have:

Theorem (3.1). In an F^3, a special vector field of first kind is such that scalars B and D satisfy equations (3.5) and (3.6).

Taking h-covariant derivative of equation (3.1), we get $\varphi_{j/k} = -h_{kj}$, showing that $\varphi_{j/k}$ is symmetric in j and k, which implies $\varphi_{j/k/h} - \varphi_{j/h/k} = 0$ or alternatively

$$K^t_{jhk}\varphi_t + (\Delta_t \varphi_j)\, K^t_{phk} y^p = 0. \qquad \ldots (3.7)$$

Taking v-covariant derivative of (3.3) we get

$$A_{//j} = L^{-1}\varphi_j,\ B_{//j} = (C_{(1)}\, B - C_{(2)}\, D - L^{-1}A)\, m_j + (C_{(3)}\, D - C_{(2)}\, B)\, n_j + L^{-1}\, D\, v_j,$$

$$D_{//j} = (C_{(3)}\, D - C_{(2)}\, B)\, m_j + ((C_{(3)}\, B + C_{(2)}\, D - L^{-1}\, A)\, n_j - L^{-1}\, B\, v_j,$$

which show that

$A_{//j} l^j = 0$, $B_{//j} l^j = 0$, $D_{//j} l^j = 0$, $A_{//j} m^j = L^{-1} B$, $B_{//j} m^j = C_{(1)}B - C_{(2)}D - L^{-1}(A - D\, v_{2)32})$, $D_{//j} m^j = (C_{(3)}D - C_{(2)}B) - L^{-1}D\, v_{2)32}$, $A_{//j} n^j = L^{-1}D$, $B_{//j} n^j = (C_{(3)}D - C_{(2)}B) + L^{-1}\, D v_{2)33}$, $D_{//j} n^j = C_{(3)}B + C_{(2)}D - L^{-1}(A + B\, v_{2)33})$

Taking v-covariant derivative of (3.1), we can obtain on simplification

$$\varphi_{j//k} = m_j m_k\, (C_{(1)}\, B - C_{(2)}D) + n_j n_k\, (C_{(3)}B + C_{(2)}D)$$
$$+ (m_j n_k + m_k n_j)\, (C_{(3)}D - C_{(2)}B) - L^{-1}(\varphi_k l_j + A\, h_{jk}), \qquad \ldots (3.8)$$

which leads to

$$\varphi_{j//k} - \varphi_{k//j} = L^{-1}(\varphi_j l_k - \varphi_k l_j) \qquad \ldots (3.9)$$

Hence we have:

Theorem (3.2). In a three dimensional Finsler space F^3, a special vector field of first kind, X^i (x) is such that $\varphi_{j/k}$ is symmetric in j and k, while $\varphi_{j//k}$ is non-symmetric in j and k and satisfies equation (3.9).

Multiplying equation (1.3) by X^i**,** we get

$$X^i C_{ijk} = (C_{(1)} B - C_{(2)} D)\, m_j m_k + (C_{(3)} D - C_{(2)} B)\,(m_j n_k + m_k n_j) + (C_{(3)} B + C_{(2)} D)\, n_j n_k. \quad \ldots (3.10)$$

In case X^i (x) is a concurrent vector field in F^3, we have Rastogi and Dwivedi [3] $X^i C_{ijk} = \alpha\, L^{-1} h_{jk}$. Now comparing equation (3.10) with this value we get

$$\alpha\, L^{-1} = C_{(1)} B - C_{(2)} D = C_{(3)} B + C_{(2)} D \text{ and } C_{(3)} D = C_{(2)} B \quad \ldots (3.11)$$

From equation (3.11), we can obtain

$$(C_{(1)} - C_{(3)})\, C_{(3)} = 2 C_{(2)}^{\,2} \quad \ldots (3.12)$$

Hence we have:

Theorem (3.3). In a Finsler space F^3, if X^i (x) is both a special vector field of first kind and a concurrent vector field, it satisfies $2\alpha = LBC$ and other coefficients in torsion tensor satisfy (3.12).

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