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A STUDY OF CERTAIN NEW CURVES IN AN EUCLIDEAN SPACE OF THREE DIMENSIONS-III

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Analogous to λ_0-curves and λ_N-curves defined by Bajpai and Rastogi [5, 6], in this paper we have defined a new curve called λ_R-curve, which is such that its rectifying plane contains the vector $d^3\lambda^i/ds^3$, where λ^i are the contra-variant components of a unit vector in the direction of the line 1 of the congruence passing through a point *P*. We have studied some curvature properties of this curve in an Euclidean space of three dimensions visa-a-vis other well-known curves.

PRELIMINARIES

Let $S : x^i = x^i (u^\alpha)$, (i = 1, 2, 3 and α = 1, 2), be the surface of reference of a rectilinear congruence, a line 1 of which is given by the direction cosines

$$\lambda^i = \lambda^i (u^\alpha),\ \lambda^i.\lambda^I = 1. \qquad \ldots (1.1)$$

We assume that x^i and λ^i are continuous along with their partial derivatives up to the required order. At any point $P (x^i)$ of S, λ^i is expressible as [3]

$$\lambda^i = p^\alpha x^i,_{\alpha} + q X^i, \qquad \ldots (1.2)$$

where p^α are the contra-variant components of a vector is S at P and q is a scalar function, X^i are the direction cosines of the normal to S at P and $x^i,_\alpha$ denotes the covariant derivative of x^i with respect to u^α based on the fundamental tensor of S, $g_{\alpha\beta} = x^i,_\alpha . x^i,_\beta$.

The Gauss and Weingarten equations Eisenhart [2] are given by $x^i,_{\alpha\beta} = d_{\alpha\beta} X^i$, $X^i,_\alpha = - d_\alpha^\delta x^i,_\delta$, where $d_{\alpha\beta}$ is the second fundamental tensor of the surface S.

Let us consider a curve C: $x^i = x^i (s)$ on S, then the intrinsic derivative of x^i, $d x^i/d s$ and $d^2 x^i/ds^2$ are expresses as

$$x'^i = d x^i/d s = x^i,_\alpha u'^\alpha,\ x''^i = d^2 x^i/ds^2 = \rho^\alpha x^i,_\alpha + X^i k_n = k\beta^I \qquad \ldots (1.3)$$

and

$$x'''^i = d^3x^i/ds^3 = (\rho^\alpha,_\beta - k_n d_{\beta\theta} g^{\theta\alpha}) x^i,_\alpha u'^\beta + (k_n,_\beta + \rho^\alpha d_{\alpha\beta}) X^i u'^\beta, \qquad \ldots (1.4)$$

where, primes indicate the differentiation with respect to are-length s, p^α are the components of the geodesic curvature vector of the curves C and k_n is the normal curvature of the surface in the direction of curves C [2].

SOME INTRINSIC DERIVATIVES

If we consider a vector λ^i in the direction of the curves of the congruence-λ, following intrinsic derivatives can be obtained:

$$\lambda'^i = d\,\lambda^i/ds = \lambda^i,_{\alpha} u'^{\alpha} = (\mu^{\gamma}_{\alpha}\, x^i,_{\gamma} + v_{\alpha} X^i)\, u'^{\alpha} \quad \ldots (2.1)$$

where $$\mu^{\gamma}_{\alpha} = p^{\gamma},_{\alpha} - q\, d_{\alpha\beta}\, g^{\beta\gamma},\ \nu_{\alpha} = q,_{\alpha} + p^{\beta}\, d_{\alpha\beta}. \quad \ldots (2.2)$$

For a normal congruence $\mu^{\gamma}_{\alpha} = -\,d\;^{\gamma}_{\alpha}$ and $v_{\alpha} = 0$, while for a congruence formed by tangents to a one parameter family of curves $\mu^{\gamma}_{\alpha} = p^{\gamma},_{\alpha}$ and $v_{\alpha} = p^{\beta}\, d_{\alpha\beta}$. We know that $\lambda^i.\lambda^i = 1$, therefore we can obtain $\lambda^i.\,\lambda^i,_{\alpha} = 0$, which gives $p_{\gamma}\mu^{\gamma}_{\alpha} + q\, v_{\alpha} = 0$.

Differentiating $\lambda^i,_{\alpha}$ covariantly with respect to u'^{β}, we get

$$\lambda^i_{,\alpha\beta} = M^{\gamma}_{\alpha\beta}\, x^i_{,\gamma} + N_{\alpha\beta}\, X^i, \quad \ldots (2.3)$$

where $$M^{\gamma}_{\alpha\beta} = \mu^{\gamma}_{\alpha,\beta} - v_{\alpha} d_{\beta\theta}\, g^{\theta\gamma},\ N_{\alpha\beta} = v_{\alpha,\beta} + \mu^{\gamma}_{\alpha} d_{\gamma\beta}. \quad \ldots (2.4)$$

The intrinsic derivative of $d\lambda^i/ds$, represented by λ''^i along C can be obtained as follows:

$$\lambda''^i = (M^{\gamma}_{\alpha\beta}\, u'^{\alpha} u'^{\beta} + \mu^{\gamma}_{\alpha} u''^{\alpha})\, x^i,_{\gamma} + (N_{\alpha\beta}\, u'^{\alpha} u'^{\beta} + v_{\alpha}\, u''^{\alpha})\, X^i. \quad \ldots (2.5)$$

such that $$p_{\gamma}\, M^{\gamma}_{\alpha\beta} + q\, N_{\alpha\beta} + \mu_{\alpha\delta}\mu^{\delta}_{\beta} + v_{\alpha} v_{\beta} = 0. \quad \ldots (2.6)$$

From equation (2.3) we can get

$$\lambda^i,_{\alpha\beta\gamma} = (M^{\theta}_{\alpha\beta,\gamma} - N_{\alpha\beta}\, d_{\gamma\delta}\, g^{\delta\theta})\, x^i,_{\theta} + X^i\, (M^{\theta}_{\alpha\beta}\, d_{\theta\gamma} + N_{\alpha\beta,\gamma}). \quad \ldots (2.7)$$

such that $$q\ \{M^{\theta}_{\alpha\beta}\, (q\, d_{\theta\gamma} + \mu_{\theta\gamma} - p_{\theta,\gamma}) + M^{\theta}_{\beta\gamma}\mu_{\theta\alpha} - M_{\beta\theta,\gamma}\mu^{\theta}_{\alpha}\}$$
$$- \mu^{\theta}_{\alpha}\mu^{\varphi}_{\beta} d_{\theta\gamma} p_{\varphi} - N_{\alpha\beta}\, (\mu^{\theta}_{\gamma}\, p_{\theta} + q\, p^{\theta} d_{\theta\gamma} + qq,_{\gamma}) = 0. \quad \ldots (2.8)$$

The intrinsic derivative of $d^2\lambda i/ds^2$ along C which is represented by λ'''^i can be obtained in the following form

$$\lambda'''^i = (x^i,\gamma\, A^{\gamma} + B\, X^i), \quad \ldots (2.9)$$

where

$$A^{\gamma} = [\mu^{\gamma}_{\alpha} u'''^{\alpha} + u''^{\alpha} u'^{\beta}\, (2M^{\gamma}_{\alpha\beta} + M^{\gamma}_{\beta\alpha}) + u'^{\alpha} u'^{\beta} u'^{\delta} (M^{\gamma}_{\alpha\beta,\delta} - N_{\alpha\beta}\, d^{\gamma}_{\delta})] \quad \ldots (2.10\text{ a})$$

and $$B = [v_{\alpha} u'''^{\alpha} + u''^{\alpha} u''^{\beta}\, (2N_{\alpha\beta} + N_{\beta\alpha}) + u'^{\alpha} u'^{\beta} u'^{\delta}\, (d_{\gamma\delta}\, M^{\gamma}_{\alpha\beta} + N_{\alpha\beta,\delta})]. \quad \ldots (2.10\text{ b})$$

λ_R-CURVES

Definition 3.1. A curve C in an Euclidean space of three dimensions shall be called λ_R-curve, if the vector λ''' lies in the rectifying plane at the point.

The differential equation of the rectifying plane to a curve C at a point P is [2]:

$$('x^i - x^i)\, (\rho^{\alpha} x^i,_{\alpha} + X^i k_n) = 0. \quad \ldots (3.1)$$

For C to be a λ_R-curve $'x^i = x^i + t\,\lambda'''$ must satisfy (2.1) for all t. Using this and (2.9) in (3.1), we get on simplification

$$A^{\alpha}\rho_{\alpha} + B\, k_n = 0, \quad \ldots (3.2)$$

as the differential equation of a λ_R-curve.

Remarks. In an earlier paper Rastogi and Bajpai [4], while studying the properties of the vector λ'''^i, obtained equation (3.2) as the differential equation of a Super Darboux curve.

Alternatively, let us suppose that α^i, β^i and γ^i be respectively the direction cosines of unit tangent, principal normal and binormal to a λ_R-curve C, then we can express λ'''^i as follows:

$$\lambda'''^i = a\,\alpha^i = b\,\gamma^i, \quad \ldots (3.3)$$

where a and b are arbitrary constants to be determined.

Let ψ be the angle between the vectors α^i and λ'''^i, then for $D\,|\lambda'''^i| = 1$, we can obtain

$$a = D^{-1} \cos\psi,\ b = \pm D^{-1} \sin\psi. \qquad \dots (3.4)$$

Substituting in equation (3.3), from equation (2.9) and (3.4), together with

$$\tau\gamma^i = -(d\,\beta^i/ds + k\alpha^i), \qquad \dots (3.5)$$

we obtain on simplification

$$x^i{}_{,\gamma}A^\gamma + B\,X^i = D^{-1}\, x^i{}_{,\gamma}[u'^\gamma \cos\psi \pm \tau^{-1} \sin\psi\ \{\rho^\gamma\, k'\, k^{-2} - k\, u'^\gamma - k^{-1} (\rho^\gamma{}_{,\beta} - k_n\, d^\gamma{}_\beta)\, u'\beta\}] \pm (D\,\tau\, k)^{-1}\, (k'\, k^{-1} k_n - d_{\alpha\beta}\rho^\alpha u'^\beta - k_n')\, X^i \sin\psi, \quad \dots (3.6)$$

which easily leads to

$$\eta^\gamma \equiv A^\gamma - D^{-1}\, [u'^\gamma \cos\psi \pm \tau^{-1} \sin\psi\ \{\rho^\gamma\, k'\, k^{-2} - k\, u'^\gamma - k^{-1}\, (\rho^\gamma{}_{,\beta} - k_n\, d^\gamma{}_\beta)\, u'\beta\}] = 0 \qquad \dots (3.7a)$$

And $$B = \pm\, (D\tau\, k)^{-1}\, (k'\, k^{-1} k_n - d_{\alpha\beta}\rho^\alpha u'^\beta - k_n') \sin\psi, \qquad \dots (3.7b)$$

where η^γ shall be called the λ_R-curvature vector of a curve C.

Equation (3.7a) represents analternative from of differential equation of λ_R-curves.

In case the vectors α^i and λ'''^i are parallel, *i.e.*, ψ =0, equations (3.7a) and (3.7b) respectively reduce to $\eta^\gamma = 1\ A^\gamma - D^{-1}\ u'^\gamma = 0$ and $B = 0$, *i.e.* equation (3.2) will imply $A^\alpha\rho_\alpha = 0$. Also $B = 0$, represents the differential equation of generalized Darboux curves [4]. Hence we have:

Theorem 3.1. In an Euclidean space of three dimensions, if the vector λ'''^i is parallel to α^i, λ_R-curve satisfies the differential equation $A^\gamma - D^{-1}\ u'^\gamma = 0$ or $A^\alpha\rho_\alpha = 0$ and it reduces to a generalized Darboux curve.

λ_R-CURVATURE

Let K_R represents the λ_R-curvature of a curve C, *i.e.*, the magnitude of the vector with contravariant components η^γ, then we can obtain after some lengthy calculation

$$K^2_R = A^\gamma A_\gamma - 2D^{-1}\, [A_\gamma\, u'^\gamma\, (\cos\psi \pm k\,\tau^{-1} \sin\psi) \pm (k\,\tau)^{-1}\ \{\sin\psi\ (k^{-1}\, k'\, A_\gamma \rho^\gamma - A_\gamma\rho'^\gamma + k_n\, d_{\gamma\beta}\, A^\gamma u'^\beta) - (1/2)\, D^{-1} \sin 2\psi\ (k_g{}^2 + u'^\gamma\rho'_\gamma)\}] + D^{-2} [\cos^2\psi + \tau^{-2} \sin^2\psi\ \{k'^2\, k^{-4} k_g{}^2 + k^{-2}\rho'^\gamma\rho'_\gamma - k^2 - 2k_n{}^2 + 2u'^\gamma\rho'_\gamma - 2k^{-3}k'\, (\rho'^\gamma\rho'_\gamma - k_n d_{\gamma\beta}\, u'^\gamma\rho^\beta) + k^{-2} k_n d_{\gamma\beta}\, u'^\beta\, (d^\gamma{}_\theta u'^\theta - 2\rho'^\gamma)\}], \quad \dots (4.1)$$

where $k_g{}^2 = k^2 - k^2_n$. Thus we have

Theorem 4.1. In a three-dimensional Euclidean space λ_R-curvature K_R of a curve C is given by (4.1) and it vanishes identically for a λ_R-curve.

For $\psi = 0$, equation (4.1) reduces to

$$K^2_R = A^\gamma A_\gamma - 2D^{-1} A_\gamma\, u'^\gamma + D^{-2}, \qquad \dots (4.2)$$

which implies.

Theorem 4.2. In a three-dimensional Euclidean space if the vector λ'''^i is parallel to α^i, λ_R-curvature K_R of a curve C is given by (4.2).

Curvature of a λ_R-curve

Let α^i, β^i and γ^i be respectively the direction cosines of unit tangent, principal normal and binormal to a curve C, then for a λ_R-curve we can express γ^i in the following form:

$$\gamma^i = p\,\alpha^i + q\lambda'''^i, \quad \dots (5.1)$$

where p and q are arbitrary constants to be determined.

Let ψ be the angle between the vectors α^i and λ'''^i, then for $D\,|\lambda'''^i| = 1$

$$p = -\cot\psi,\ q = D\operatorname{cosec}\psi. \quad \dots (5.2)$$

From equations (5.1) and (5.2), we get

$$\gamma^i = \operatorname{cosec}\psi\,(D\lambda'''^i - \alpha^i\cos\psi).$$

Comparing equations (3.5) and (5.3), we get on simplification

$$k^2 = \rho_{\alpha,\beta}\,u'^\alpha u'^\beta - k_n^2.$$

Hence we have:

Theorem 5.1.- The curvature of a λ_R-curve C, in an Euclidean space of three dimensions is given by (5.4) and this curvature will vanish if and only if the curve satisfies

$$(\rho_{\alpha,\beta} - d_{\alpha\gamma}d_{\beta\delta}\,u'^\delta)\,u'^\alpha,\ u'^\beta = 0. \quad \dots (5.5)$$

From equation (1.3) we can easily obtain

$$k^2 = \rho^\alpha{}_{,}\,\rho_\alpha + k^2_n, \quad \dots (5.6)$$

therefore, comparing equations (5.4) and (5.6) we get

$$k^2 = (1/2)\,(\rho_{\alpha,\beta}\ u'^\alpha u'^\beta + \rho^\alpha\rho_\alpha) \quad \dots (5.7)$$

and

$$k_n^2 = (1/2)\,(\rho_{\alpha,\beta}\ u'^\alpha u'^\beta - \rho^\alpha\rho_\alpha) \quad \dots (5.8)$$

Hence we have :

Theorem 5.2. The curvature of a λ_R-curve C, in an Euclidean space of three dimensions satisfies equation (5.7) while the normal curvature of the surface is given by equation (5.8).

Substituting the values of α^i, β^i and γ^i, with the help of equations (5.3) and (5.4), we can obtain

$$\tau = -(BD)^{-1}\sin\psi\,\{(k_n/k)' + k^{-1}\,d_{\alpha\beta}\rho^\alpha\,u'^\beta\}. \quad \dots (5.9)$$

If we assume that $d_{\alpha\beta}\rho^\alpha\,u'^\beta = 0$, equation (5.9) gives $\tau = -(BD)^{-1}\sin\psi\ \{(k_n/k)'$ and conversely. Hence we have:

Theorem 5.3. The necessary and sufficient condition for λ_R-curves to be conjugate to their curvature vectors, is given by $\tau = -(BD)^{-1}\sin\psi\,(k_n/k)'$.

If we assume $\tau = 0$, $d_{\alpha\beta}\rho^\alpha\,u'^\beta = -k\,(k_n/k)'$. Conversely, if $d_{\alpha\beta}\rho^\alpha\,u'^\beta = -k\,(k_n/k)'$, we get $\tau = 0$. Hence we have:

Theorem 5.4. The necessary and sufficient condition for a λ_R-curve to have vanishing torsion is given by $d_{\alpha\beta}\rho^\alpha\,u'^\beta = -k\,(k_n/k)'$.

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