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A CLASS OF BE – ALGEBRAS

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In this paper we introduce a method to form a BE – algebra of all fuzzy sets defined on an universe S. We prove here that this BE - algebra is commutative and transitive but it is not self distributive.

Introduction

The concept of a BE – algebra was introduced by H. S. Kim and Y. H. Kim in 2006. Since then several related concepts have been studied by different authors.

Definition (1.1) : A system $(X; *, 1)$ of type (2, 0) consisting of a non-empty set X, a binary operation "*" and a fixed element 1 is called a BE–algebra if the following axioms are satisfied :

1. (BE 1) $x * x = 1$
2. (BE 2) $x * 1 = 1$
3. (BE 3) $1 * x = x$
4. (BE 4) $x * (y * z) = y * (x * z)$

for all $x, y, z \in X$.

Example (1.2) : A simplest example of a BE–algebra is as follows:-

Let $X = \{1, 0\}$ and let a binary operation '*' be defined as

Table 1

*	0	1
0	1	1
1	0	1

Then $(X; *, 1)$ is a BE–algebra.

Proposition (1.3) : In a BE – algebra $(X; *, 1)$ the following hold:

(a) $x * (y * x) = 1$,

(b) $x * ((x * y) * y) = 1$

for any $x, y \in X$.

Definition (1.4) : A BE–algebra $(X; *, 1)$ is said to be

(a) Commutative, if

$$(x * y) * y = (y * x) * x \quad \forall \quad x, y \in X;$$

(b) Self distributive, if for any $x, y, z \in X$

$$x * (y * z) = (x * y) * (x * z);$$

(c) transitive, if for any $x, y, z \in X$

$$(y * z) * ((x * y) * (x * z)) = 1.$$

MAIN RESULTS

Let S be a non-empty set and let X be set of all functions defined on S with values in [0, 1], *i.e.*; X is the set of all fuzzy sets on S.

Let 1^* and 0^* be the functions defined on S as

$$1^* (x) = 1 \text{ and } 0^* (x) = 0 \text{ for all } x \in S. \qquad \ldots (2.1)$$

Also for $f, g \in X$, we define $f = g$ iff $f(x) = g(x)$ for all $x \in S$.

We prove the following results.

Lemma (2.1). For $f, g \in X$ we define

$$(fog)(x) = \min \{f(x), g(x)\} + 1 - f(x) \qquad \ldots (2.2)$$

$$= (f \wedge g)(x) + 1 - f(x)$$

Then 'o' is a binary operation in X.

Proof : For $x \in S$, let $\min \{f(x), g(x)\} = f(x)$. Then

$$(fog)(x) = f(x) + 1 - f(x) = 1$$

Again let $\min \{f(x), g(x)\} = g(x) = s$ and let $f(x) = t$. Then

$$(fog)(x) = s + 1 - t = 1 - (t - s) < 1$$

So 'o' defines a binary operation on S.

Lemma (2.2) (a) If $f(x) < g(x)$ on S then $fog = 1^*$.

(b) If $g(x) < f(x)$ on S then $(fog)(x) = 1 + g(x) - f(x)$.

Proof : (a) For $x \in S$, $\min \{f(x), g(x)\} = f(x)$ and so

$$(fog)(x) = f(x) + 1 - f(x) = 1 = 1^* (x)$$

$$\Rightarrow \quad fog = 1^*.$$

(b) If $g(x) < f(x)$ then

$$(fog)(x) = g(x) + 1 - f(x)$$

$$= 1 - (f(x) - g(x))$$

$$< 1.$$

Theorem (2.3) : The system $(X; o, 1^*)$ is a BE–algebra with zero element 0^* where binary operation ''o'' is defined by (2.2).

Proof : For any $x \in S$ and $f \in X$ we have

(BE1) $(f o f)(x) = f(x) + 1 - f(x) = 1 = 1^* (x)$

$\Rightarrow \quad f o f = 1^*$;

(BE2) $(f \ o \ 1^*)(x) = f(x) + 1 - f(x) = 1 = 1^* (x)$

$\Rightarrow \quad f \, o \, 1^* = 1^*$;

(BE3) $(1^* \, o \, f)(x) = f(x) + 1 - 1 = f(x) \Rightarrow 1^* \, o \, f = f$.

To prove (BE4), *i.e.*, $f \, o \, (g \, o \, h) = g \, o \, (f \, o \, h)$ for $f, g, h \in X$,

we consider the following cases.

Case (a) : Let $f(x) < g(x) < h(x)$ on $A \subseteq S$.

Then by lemma (2.2) (a) we have $f \, o \, h = 1^*$ and $g \, o \, h = 1^*$

This gives $g \, o \, (f \, o \, h) = 1^*$ and $f \, o \, (g \, o \, h) = 1^*$ by (BE2).

So $\quad f \, o \, (g \, o \, h) = g \, o \, (f \, o \, h)$ on A.

Case (b) : Let $g(x) < f(x) < h(x)$ on $B \subseteq S$.

Then $f \, o \, (g \, o \, h) = g \, o \, (f \, o \, h)$ on B as in case (a).

Case (c) : Let $f(x) < h(x) < g(x)$ on $C \subseteq S$.

Then $f \, o \, h = 1^*$ by lemma (2.2) (a) and $g o \, (f \, o \, h) = 1^*$ on C by (BE2)

Also for $x \in C$, $(g \, o \, h)(x) = h(x) + 1 - g(x)$.

So $\quad \min \{f(x), h(x) + 1 - g(x)\} = f(x)$ on C.

Thus for $x \in C$, $(f \, o \, (g \, o \, h))(x) = f(x) + 1 - f(x) = 1 = 1^*(x)$

$\Rightarrow \quad f \, o \, (g \, o \, h) = 1^*$.

This proves that $f \, o \, (g \, o \, h) = g \, o \, (f \, o \, h)$ on C.

Case (d) : Let $g(x) < h(x) < f(x)$ on $D \subseteq S$

Then $g \, o \, (f \, o \, h) = f \, o \, (g \, o \, h)$ on D follows from case (c).

Case (e) : Let $h(x) < f(x) < g(x)$ on $E \subseteq S$

Then for $x \in E$, $(g \, o \, h)(x) = h(x) + 1 - g(x)$

and $\quad (f \, o \, h)(x) = h(x) + 1 - f(x)$.

Now either

(α) $f(x) < h(x) + 1 - g(x) \Rightarrow g(x) < h(x) + 1 - f(x)$

or $\quad (\beta)$ $h(x) + 1 - g(x) < f(x) \Rightarrow h(x) + 1 - f(x) < g(x)$.

In case (α), we have

$(f \, o \, (g \, o \, h))(x) = f(x) + 1 - f(x) = 1$

and $\quad (g \, o \, (f \, o \, h))(x) = g(x) + 1 - g(x) = 1$

In case (β), we have

$$(f \, o \, (g \, o \, h))(x) = h(x) + 1 - g(x) + 1 - f(x)$$
$$= 2 + h(x) - g(x) - f(x)$$

and $\quad (g \, o \, (f \, o \, h))(x) = 2 + h(x) - g(x) - f(x)$

So in both cases (α) and (β) we have

$f \, o \, (g \, o \, h) = g \, o \, (f \, o \, h)$ on E.

Case (f) : Let $h(x) < g(x) < f(x)$ on $F \subset S$.

In this case also we have

$$f\ o\ (g\ o\ h) = g\ o\ (f\ o\ h)$$

Since S is disjoint union of A, B, C, D, E and F; we see that

$$f\ o\ (g\ o\ h) = g\ o\ (f\ o\ h) \text{ in all cases.}$$

Hence $(X;\ 0, *)$ is a BE–algebra.

Note (2.4) : X also contains 0^* satisfying $0^*\ o\ f = 1^*$ for all $f \in X$.

Definition (2.5) : The complement of a function f, denoted as f^c, is defined as

$$f^c\ (x) = (f\ o\ 0^*)\ (x) = 0 + 1 - f(x) = 1 - f(x) \text{ for all } x \in S.$$

Lemma (2.6) : We have $(f^c)^c = f$.

Proof : Let $f^c = g$. Then

$$g^c\ (x) = 1 - g\ (x) = 1 - (1 - f(x) = f(x)$$

for all $x \in S$. This proves that $g^c = f$, *i.e.*, $(f^c)^c = f$.

Lemma (2.7) : $f\ o\ g = g^c\ o\ f^c$

Proof : Let $f(x) \leq g(x)$ on S_1 and $g(x) < f(x)$ on S_2.

For $x \in S_1$ we have, $(f\ o\ g)\ (x) = 1$, $f^c\ (x) = 1 - f(x)$, $g^c\ (x) = 1 - g\ (x)$

and $(g^c\ \text{o}\ f^c)\ (x) = 1$, since $g^c\ (x) \leq f^c\ (x)$.

This gives $(f\ o\ g)\ (x) = (g^c\ \text{o}\ f^c)\ (x)$ for all $x \in S_1$.

Again for $x \in S_2$ we have

$$(f\ o\ g)\ (x) = g\ (x) + 1 - f(x), \qquad \ldots(2.3)$$

$$f^c\ (x) = 1 - f(x),\ g^c\ (x) = 1 - g\ (x).$$

Since $f^c\ (x) < g^c\ (x)$ we have

$$\begin{aligned}(g^c\ o\ f^c)\ (x) &= f^c\ (x) + 1 - g^c\ (x)\\ &= 1 - f(x) + 1 - 1 + g(x)\\ &= g\ (x) + 1 - f(x)\end{aligned}$$

From (2.3) and (2.4) we have

$$(f\ o\ g)\ (x) = (g^c\ o\ f^c)\ (x) \quad \text{on } S_2.$$

Since S is disjoint union of S_1 and S_2 we have the result.

Theorem (2.8) : A set X of function defined on S into $[0, 1]$ is a BE–algebra under a binary operation 'o' defined by (2.2) with zero element 0^* iff it is closed with respect to complement and sum of functions $f(x)$ and $g(x)$ provided $f(x) + g(x) \leq 1$.

Proof : Suppose that X is a BE – algebra under binary operation 'o' with zero element 0^*. Then X contains 1^*. Also $(0^*)^c = 1^*$ and $(1^*)^c = 0^*$. If $X = \{0^*, 1^*\}$ then it is closed w. r. t. complement and sum of functions.

We assume that $0^* \neq f \neq 1^*$ is an element of X. Then

$$f \; o \; 0^* = f^c \in X. \text{ But}$$

$(f \, o \, 0^*)(x) = \min \{f(x), 0\} + 1 - f(x)$

$= 0 + 1 - f(x) = f^c(x)$ implies that $f^c \in X$. So X is closed w.r.t. complement.

Let $f, g \in X$ and $f(x) + g(x) \leq 1$

Then $g(x) \leq 1 - f(x)$ on X.

Now $f^c \; o \; g \in X \Rightarrow (f^c \; o \; g)(x) = \min \{1 - f(x), g(x)\} + 1 - f^c(x)$.

$= g(x) + 1 - 1 + f(x)$

$= g(x) + f(x) = (g + f)(x)$

$\Rightarrow g + f \in X$.

So X is closed w. r. t addition.

Conversely, suppose that X is closed w. r. t to complement and sum of functions f and g provided $f(x) + g(x) \leq 1$.

Let $f, g \in X$ and $f(x) + g(x) \leq 1$ for all $x \in S$.

Also $(f \, o \, g)(x) = \min \{f(x), g(x)\} + 1 - f(x)$

$= 1$ or $g(x) + 1 - f(x)$,

according as $f(x) < g(x)$ or $g(x) \leq f(x)$.

Since $g(x) \leq f(x) \Rightarrow g(x) + 1 - f(x) \leq 1$, according to given condition $(f \, o \, g) \in X$. Also other conditions of a BE – algebra can be proved as in theorem (2.3).

Hence the result.

Theorem (2.9) : The BE – algebra $(X; 0, 1^*)$ is transitive .

Proof : To examine the equality

$$(g \, o \, h) \, o \, ((f \, o \, g) \, o \, (f \, o \, h)) = 1^*. \; f, g, h \in X.$$

It is easy to see that whenever $f \, o \, h = 1^*$

Then $(g \, o \, h) \, o \, ((f \, o \, g) \, o \, (f \, o \, h)) = 1^*$

We consider the cases discussed in the proof of theorem (2.3).

In cases (a), (b) and (c) $f(x) < h(x)$

$\Rightarrow f \, o \, h = 1^*$ [Lemma (2.2(a))]

In case (d) we have $(f \, o \, h)(x) = h(x) + 1 - f(x)$

and $(f \, o \, g)(x) = g(x) + 1 - f(x)$.

Now $g(x) < h(x) \Rightarrow (f \, o \, g)(x) < (f \, o \, h)(x)$.

$\Rightarrow$ $((f \, o \, g) \, o \, (f \, o \, h))(x) = 1$ *i.e.*, $(f \, o \, g) \, o \, (f \, o \, h) = 1^*$

So $(g \, o \, h) \, o \, ((f \, o \, g) \, o \, (f \, o \, h)) = 1^*$.

In case (e), $(f \, o \, h)(x) = h(x) + 1 - f(x)$

and $(f \, o \, g) \, o \, (f \, o \, h) = (f \, o \, h)$, since $(f \, o \, g) = 1^*$.

Also $(g \, o \, h)(x) = h(x) + 1 - g(x)$.

Since $f(x) < g(x)$, we see that $(g o h)(x) < (f o h)(x)$

So $(g o h) o (f o h) = 1^*$ on E.

This gives $(g o h) o ((f o g) o (f o h)) = (g o h) o (f o h) = 1^*$ on E.

In case (f)

$$(f o g)(x) = h(x) + 1 - f(x)$$

and

$$(f o h)(x) = h(x) + 1 - f(x)$$

Since $h(x) < g(x)$

$$(f o h)(x) < (f o g)(x)$$

So $[(f o g) o (f o h)](x) = (f o h)(x) + 1 - (f o g)(x)$

$$= h(x) + 1 - f(x) + 1 - g(x) - 1 + f(x)$$

$$= h(x) + 1 - g(x)$$

Also $(g o h)(x) = h(x) + 1 - g(x)$

So $(g o h) o ((f o g) o (f o h)) = 1^*$ on F

Thus we see that $(g o h) o ((f o g) o (f o h)) = 1^*$ is true in all the cases.

Hence $(X; o, 1^*)$ is transitive.

Note (2. 10) : (a) $(X; o\ 1^*)$ is not self distributive.

We consider the constant functions

$$f(x) = 0.3,\ g(x) = 0.4 \text{ and } h(x) = 0.2$$

for all $x \in X$.

Then $(g o h)(x) = 0.2 + 1 - 0.4$

$$= 0.8$$

and so,

$$(f o (g o h))(x) = 0.3 + 1 - 0.3 = 1$$

Again $(f o g)(x) = 1$

and $(f o h)(x) = 0.2 + 1 - 0.3 = 0.9$

This gives $((f o g) o (f o h))(x) = 0.9 + 1 - 1 = 0.9$

Hence $f o (g o h) \neq (f o g) o (f o h)$.

Theorem (2.11) : $(X; 0, 1^*)$ is commutative

Proof : Let $f, g \in X$. Let $f(x) < g(x)$ on A, $f(x) = g(x)$ on B.

and $g(x) < f(x)$ on C.

For $x \in A$, we have

$$(f o g) = 1^* \text{ and so } (f o g) o g = g.$$

Again $(g o f)(x) = f(x) + 1 - g(x)$

$$> f(x)$$

So $((g o f) o f)(x) = f(x) + 1 - f(x) - 1 + g(x) = g(x)$

This gives $(f o g) o g = (g o f)$ of on A.

Similarly we can prove that $(f o g) o g = (g o f)$ of on C

Also $(f\ o\ g)\ o\ g = (g\ o\ f)$ of on B

Hence $(f\ o\ g)\ o\ g = (g\ o\ f)$ of on X.

Hence $(X; 0, 1^*)$ is commutative.

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