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SOME CLASSES OF PRIME LEBELING OF GRAPHS

S. MEENA AND J. NAVEEN

Associate Professor of Mathematics, Government Arts College, C. Mutlur, Chidambaram, Tamil Nadu, India

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A Graph *G* with *n* vertices is said to admit prime labeling if its vertices can be labeled with distinct positive integers not exceeding n such that the labels of each pair of adjacent vertices are relatively prime. A graph *G* which admits prime labeling is called a prime graph. In this paper, we investigate the existence of prime labeling of some graphs related to comb P_n^*, crown C_n^* , Helm H_n, Gear graph G_n and Friendship graph T_n , We discuss prime labeling in the context of the graph operation namely duplication.

KEYWORDS : Graph Labeling, Prime Labeling, Duplication, Prime Graphs.

Introduction

We begin with simple, finite, connected and undirected graph *G* (*V*, *E*) with *p* vertices and *q* edges. The set of vertices adjacent to a vertex *u* of *G* is denoted by N(u). For notations and terminology we refer to Bondy and Murthy [1].

The notion of prime labeling was introduced by Roger Entringer and was discussed in a paper by Tout [6]. Two integers *a* and *b* are said to be relatively prime if their greatest common divisor is 1. Relatively prime numbers play an important role in both analytic and algebraic number theory. Many researchers have studied prime graph. Fu. H [3] has proved that the path P_n on *n* vertices is a prime graph. Deretsky *el al* [2] have prove that the cycle C_n on *n* vertices is a prime graph. Around 1980 Roger Etringer conjectured that all trees have prime labeling which is not setteled till today.

The prime labeling for planer grid was investigated by Sundaram *et al* [5], Lee.S.at.al [4] has proved that the Wheel W_n is a prime graph if and only if *n* is even.

Definition 1.1[7] Duplication of an edge $v_i v_{i+1}$ of a graph *G* produces a new graph G_1 by adding a new edge $v_i' v_{i+1}'$ in such a way that N(v_i') = N(v_i) ∪ { v_{i+1}'} − {v_{i+1}} and N(v_{i+1}') = N(v_{i+1}) ∪ { v_i'} - {v_i}.

Definition 1.2 The graph obtained by duplicating every edge by an edge of a graph *G* is called duplication of *G*.

Definition 1.3 The comb P_n^* is obtained from a path P_n by attaching a pendent edge at each vertex of the path P_n.

Definition 1.4 The crown graph C_n^* is obtained from a cycle C_n by attaching a pendent edge at each vertex of the *n*-cycle.

Definition 1.5 The Helm H_n is a graph obtained from a Wheel by attaching a pendent edge at each vertex of the n-cycle.

Definition 1.6 The Gear graph G_n is, the graph obtained from wheel $W_n = C_n + K_1$ by subdividing each edge of the n- cycle.

Definition 1.7 The Friendship graph T_n is set of n triangles having a common central vertex.

In this paper we proved that the graph obtained by duplicating every pendent edge by an edge in comb P_n^* , crown C_n^* , Helm H_n ,the graph obtained by duplicating every alternate rim edge by an edge in crown C_n^* , Helm H_n , Gear graph G_n and the graph obtained by duplicating every edge by an edge except the incident edges of central vertex in Friendship graph T_n are all prime graphs.

MAIN RESULTS

Theorem 2.1 The graph obtained by duplicating every pendent edge by an edge in comb P_n^* is prime graph.

Proof. Let $V(P_n^*) = \{ u_i , v_i / \ 1 \leq i \leq n\}$

$$E(P_n^*) = \{u_i u_{i+1} / 1 \leq i \leq n-1\} \cup \{u_i v_i / \ 1 \leq i \leq n\}$$

Let G be the graph obtained by duplicating every pendent edge by an edge in comb P_n^* and let the new edges be $u_1'v_1', u_2'v_2', \dots, u_n'v_n'$ by duplicating the pendent edges $u_1v_1, u_2v_2, \dots, u_nv_n$ respectively,

Then $V(G) = \{ u_i, v_i, u_i', v_i' / \ 1 \leq i \leq n\}$

$$E(G) = \left\{\frac{u_i u_{i+1}}{1}\right. \leq i \leq n-1\} \cup \{u_iv_i, u_i'v_i' \ / \ 1 \leq i \leq n\} \cup \{u_i u_{i+1}', u_i'u_{i+1} / 1 \leq i \leq n-1\}$$

$|V(G)| = 4n, \qquad |E(G)| = 5n-3$

Define a labeling $f : V(G) \to \{1,2,3, \dots, 4n\}$ as follows.

Let $f(u_1) = 1 , f(v_1) = 2 , f(u_1') = 3 , f(v_1') = 4$

For $2 \leq i \leq n \ and \ i \not\equiv 1(mod\ 3)$

$$f(u_i) = 4i-1, \quad f(v_i) = 4i, \quad f(u_i') = 4i-3\,, \quad f(v_i') = 4i-2,$$

For $2 \leq i \leq n$ and $i \equiv 1(mod\ 3)$

$$f(u_i) = 4i-3, \quad f(v_i) = 4i-2, \quad f(u_i') = 4i-1\,, \quad f(v_i') = 4i,$$

For $2 \leq i \leq n-1 \ and \ i \not\equiv 1(mod\ 3)$

$\gcd(f(u_i), f(v_i)) = \gcd(4i-1, 4i) = 1, \ \gcd(f(u_i'), f(v_i')) = \gcd(4i-3, 4i-2) = 1$

Suppose $i \equiv 0(mod\ 3) \quad then \ \ i+1 \equiv 1(mod\ 3)$

$$\gcd(f(u_i), f(u_{i+1})) = \gcd(4i-1\,, 4(i+1)-3) = \gcd(4i-1\,, 4i+1) = 1$$

as these two number are consecutive odd integers

$$\gcd(f(u_i), f(u_{i+1}')) = \gcd(4i-1\,, 4(i+1)-1) = \gcd(4i-1\,, 4i+3) = 1$$

$$\gcd(f(u_i'), f(u_{i+1})) = \gcd(4i-3\,, 4(i+1)-3) = \gcd(4i-3\,, 4i+1) = 1$$

as these two numbers are odd and their difference is 4 and they are not multiples of 4

Suppose $i \equiv 2(mod\ 3) \quad then \ \ i+1 \equiv 0(mod\ 3)$

$$\gcd(f(u_i), f(u_{i+1})) = \gcd(4i-1, 4(i+1)-1) = \gcd(4i-1, 4i+3) = 1$$

as these two numbers are odd and their difference is 4 and they are not multiples of 4

$$\gcd(f(u_i), f(u'_{i+1})) = \gcd(4i-1, 4(i+1)-3) = \gcd(4i-1, 4i+1) = 1$$

as these two number are consecutive odd integers

$$\gcd(f(u'_i), f(u_{i+1})) = \gcd(4i-3, 4(i+1)-1) = \gcd(4i-3, 4i+3) = 1$$

as these two numbers are odd and their difference is 6 and they are not multiples of 6

For $2 \le i \le n-1$ and $i \equiv 1(mod\ 3)$

$$\gcd(f(u_i), f(v_i)) = \gcd(4i-3, 4i-2) = 1$$

$$\gcd(f(u'_i), f(v'_i)) = \gcd(4i-1, 4i) = 1$$

$$\gcd(f(u_i), f(u_{i+1})) = \gcd(4i-3, 4(i+1)-1) = \gcd(4i-3, 4i+3) = 1$$

as these two numbers are odd and their difference is 6 and they are not multiples of 6

$$\gcd(f(u_{i-1}), f(u_i)) = \gcd(4(i-1)-1, 4i-3) = \gcd(4i-5, 4i-3) = 1$$

as these two number are consecutive odd integers

$$\gcd(f(u_{i-1}), f(u'_i)) = \gcd(4i-5, 4i-1) = 1$$

as these two numbers are odd and their difference is 4

$$\gcd(f(u_i), f(u'_{i+1})) = \gcd(4i-3), 4(i+1)-3) = \gcd(4i-3, 4i+1) = 1$$

$$\gcd(f(u'_{i-1}), f(u_i)) = \gcd(4(i-1)-3, 4i-3) = \gcd(4i-7, 4i-3) = 1$$

$$\gcd(f(u'_i), f(u_{i+1})) = \gcd(4i-1, 4i+3) = 1$$

as these two numbers are odd and their difference is 4

Thus f is a prime labeling.

Hence G is a prime graph.

Illustration 2.1

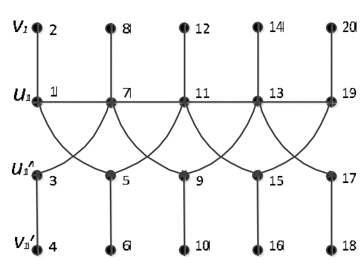

Fig. 1. Prime labeling of duplication of every pendent edge by an edge in comb P_5^*

Theorem 2.2. The graph obtained by duplicating every pendent edge by an edge in Crown C_n^* is a prime graph for all $n \ge 3$.

Proof: Let $V(C_n^*) = \{u_i, v_i \ / \ 1 \le i \le n\}$

$$E(C_n^*)=\{u_i u_{i+1} / 1 \le i \le n-1\} \cup \{u_i v_i / 1 \le i \le n\} \cup \{u_n u_1\}$$

Let G be the graph obtained by duplicating every pendent edge by an edge in Crown C_n^* and let the new edges be $u'_1 v'_1, u'_2 v'_2, \dots, u'_n v'_n$ obtained by duplicating the pendent edges $u_1 v_1, u_2 v_2, \dots, u_n v_n$ respectively, Then $V(G) = \{ u_i, v_i, u_i', v_i' / \ 1 \le i \le n\}$

$E(G) = \{u_i\, u_{i+1}, u_i\, u_{i+1}', u_i' u_{i+1} / 1 \le i \le n-1\} \cup \{u_i v_i, u_i' v_i' \ / \ 1 \le i \le n\} \cup \{ u_n u_1, u_n\, u_1', u_n' u_1 \}$

$|V(G)| = 4n,\ |E(G)| = 5n$

Define a labeling $f : V(G) \to \{1, 2, 3, \dots, 4n\}$ as follows.

For $1 \le i \le n$ and $i \not\equiv 0 (mod\ 3)$

$f(u_i) = 4i - 3,\quad f(v_i) = 4i - 2,\quad f(u_i') = 4i - 1,\quad f(v_i') = 4i,$

For $1 \le i \le n$ and $i \equiv 0 (mod\ 3)$

$f(u_i) = 4i - 1,\quad f(v_i) = 4i,\quad f(u_i') = 4i - 3,\quad f(v_i') = 4i - 2,$

Since $f(u_1) = 1$

$\gcd\big(f(u_1), f(v_1)\big) = 1,\quad \gcd\big(f(u_1), f(u_2)\big) = 1,\quad \gcd\big(f(u_1), f(u_2')\big) = 1,$

$\gcd\big(f(u_1), f(u_n)\big) = 1,\quad \gcd\big(f(u_1), f(u_n')\big) = 1,$

$\gcd(f(u_n), f(u_1')) = \gcd(4n - 3, 3) = 1,$ for $n \not\equiv 0 (mod\ 3)$

$\gcd(f(u_n), f(u_1')) = \gcd(4n - 1, 3) = 1,$ for $n \equiv 0 (mod\ 3)$

For $2 \le i \le n - 1$ and $i \not\equiv 0 (mod\ 3)$

$\gcd\big(f(u_i), f(v_i)\big) = \gcd(4i - 3, 4i - 2) = 1$

$\gcd\big(f(u_i'), f(v_i')\big) = \gcd(4i - 1, 4i) = 1$

Suppose $i \equiv 1 (mod\ 3)$ $then$ $i + 1 \equiv 2 (mod\ 3)$

$\gcd\big(f(u_i), f(u_{i+1})\big) = \gcd(4i - 3, 4(i + 1) - 3) = \gcd(4i - 3, 4i + 1) = 1$

as these two numbers are odd and their difference is 4 and they are not multiples of 4

$\gcd\big(f(u_i), f(u_{i+1}')\big) = \gcd(4i - 3, 4(i + 1) - 1) = \gcd(4i - 3, 4i + 3) = 1$

as these two numbers are odd and their difference is 6 and they are not multiples of 6

$\gcd\big(f(u_i'), f(u_{i+1})\big) = \gcd(4i - 1, 4(i + 1) - 3) = \gcd(4i - 1, 4i + 1) = 1$

as these two number are consecutive odd integers

Suppose $i \equiv 2 (mod\ 3)$ $then$ $i + 1 \equiv 0 (mod\ 3)$

$\gcd\big(f(u_i), f(u_{i+1})\big) = \gcd(4i - 3, 4(i + 1) - 1) = \gcd(4i - 3, 4i + 3) = 1$

as these two numbers are odd and their difference is 6 and they are not multiples of 6

$\gcd\big(f(u_i), f(u_{i+1}')\big) = \gcd(4i - 3, 4(i + 1) - 3) = \gcd(4i - 3, 4i + 1) = 1$

$\gcd\big(f(u_i'), f(u_{i+1})\big) = \gcd(4i - 1, 4(i + 1) - 1) = \gcd(4i - 1, 4i + 3) = 1$

as these two numbers are odd and their difference is 4 and they are not multiples of 4

For $2 \le i \le n - 1$ and $i \equiv 0 (mod\ 3)$ then $i + 1 \not\equiv 0 (mod\ 3)$ and $i - 1 \not\equiv 0 (mod\ 3)$

$\gcd\big(f(u_i), f(v_i)\big) = \gcd(4i - 1, 4i) = 1$

$\gcd\big(f(u_i'), f(v_i')\big) = \gcd(4i - 3, 4i - 2) = 1$

as these two number are consecutive integers

$\gcd\big(f(u_i), f(u_{i+1})\big) = \gcd(4i - 1, 4(i + 1) - 3) = \gcd(4i - 1, 4i + 1) = 1$

as these two number are consecutive odd integers

$$\gcd\big(f(u_{i-1}), f(u_i)\big) = \gcd(4(i-1)-3\,, 4i-1) = \gcd(4i-7\,, 4i-1) = 1$$

as these two numbers are odd and their difference is 6 and they are not multiples of 3 and 6

$$\gcd\big(f(u_{i-1}), f(u_i')\big) = \gcd(4i-7\,, 4i-3) = 1$$

$$\gcd\big(f(u_i), f(u_{i+1}')\big) = \gcd(4i-1)\,, 4(i+1)-1) = \gcd(4i-1\,, 4i+3) = 1$$

$$\gcd\big(f(u_i'), f(u_{i+1})\big) = \gcd(4i-3\,, 4i+1) = 1$$

as these two numbers are odd and their difference is 4 and they are not multiples of 2 and 4

Thus f is a prime labeling.

Hence G is a prime graph.

Illustration 2.2

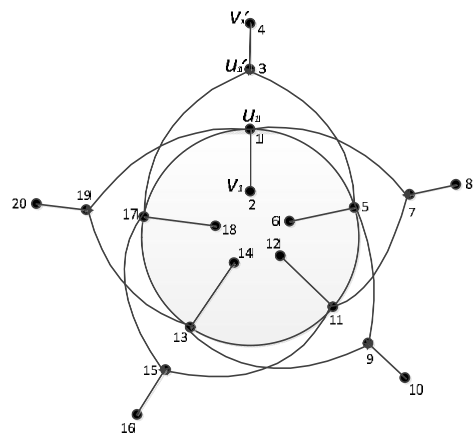

Fig. 2. Prime labeling of duplication of every pendent edge by an edge in Crown C_5^*

Theorem 2.3. The graph obtained by duplicating every pendent edge by an edge in Helm H_n is a prime graph for all n ≥ 3.

Proof. Let $V(H_n) = \{c, u_i, v_i \;/\; 1 \le i \le n\}$

$$E(H_n) = \{cu_i, u_iv_i \;/ 1 \le i \le n\} \cup \{u_i u_{i+1} / 1 \le i \le n-1\} \cup \{u_nu_1\}$$

Let G be the graph obtained by duplicating every pendent edge by an edge in Helm H_n and let the new edges be $u_1'v_1', u_2'v_2', \ldots, u_n'v_n'$ obtained by duplicating the pendent edges $u_1v_1, u_2v_2, \ldots, u_nv_n$ respectively, Then

$$V(G) = \{c, u_i, v_i, u_i', v_i' \;/\; 1 \le i \le n\}$$

$$E(G) = \{cu_i, u_iv_i, cu_i', u_i'v_i' \;/ 1 \le i \le n\} \cup \{u_i u_{i+1}, u_i u_{i+1}', u_i'u_{i+1} / 1 \le i \le n-1\} \cup \{u_nu_1, u_n u_1', u_n'u_1\}$$

$|V(G)| = 4n+1,\ |E(G)| = 7n,$

Define a labeling $f : V(G) \to \{1,2,3,\ldots,4n+1\}$ as follows

Let $f(c) = 1$, $f(u_1) = 2$, $f(v_1) = 3$, $f(u_1') = 4$, and $f(v_1') = 5$

For $2 \le i \le n$ *and* $i \not\equiv 1 (mod\ 3)$

$$f(u_i) = 4i-1, \quad f(v_i) = 4i-2, \quad f(u_i') = 4i+1, \quad f(v_i') = 4i,$$

For $2 \leq i \leq n$ and $i \equiv 1(mod\ 3)$

$$f(u_i) = 4i + 1, \quad f(u_i') = 4i - 1$$

Since $f(c) = 1$

$\gcd\big(f(c), f(u_i)\big) = 1$, $\gcd\big(f(c), f(u_i')\big) = 1$, $\gcd\big(f(u_1), f(u_n)\big) = \gcd(2, 4n - 1) = 1$.

$\gcd(f(u_n), f(u_1')) = \gcd(4n - 1, 4) = 1$, $\gcd\big(f(u_1), f(u_n')\big) = \gcd(2, 4n + 1) = 1$

For $2 \leq i \leq n - 1$ $and\ i \not\equiv 1(mod\ 3)$

$$\gcd\big(f(u_i), f(v_i)\big) = \gcd(4i - 1, 4i - 2) = 1$$

$$\gcd\big(f(u_i'), f(v_i')\big) = \gcd(4i + 1, 4i) = 1$$

Suppose $i \equiv 0(mod\ 3)$ $then$ $i + 1 \equiv 1(mod\ 3)$

$$\gcd\big(f(u_i), f(u_{i+1})\big) = \gcd(4i - 1, 4(i + 1) + 1) = \gcd(4i - 1, 4i + 5) = 1$$

as these two numbers are odd and their difference is 6 and they are not multiples of 6

$$\gcd\big(f(u_i), f(u_{i+1}')\big) = \gcd(4i - 1, 4(i + 1) - 1) = \gcd(4i - 1, 4i + 3) = 1$$

$$\gcd\big(f(u_i'), f(u_{i+1})\big) = \gcd(4i + 1, 4(i + 1) + 1) = \gcd(4i + 1, 4i + 5) = 1$$

as these two numbers are odd and their difference is 4 and they are not multiples of 4

Suppose $i \equiv 2(mod\ 3)$ $then$ $i + 1 \equiv 0(mod\ 3)$

$$\gcd\big(f(u_i), f(u_{i+1})\big) = \gcd(4i - 1, 4(i + 1) - 1) = \gcd(4i - 1, 4i + 3) = 1$$

as these two numbers are odd and their difference is 4 and they are not multiples of 4

$$\gcd\big(f(u_i), f(u_{i+1}')\big) = \gcd(4i - 1, 4(i + 1) + 1) = \gcd(4i - 1, 4i + 5) = 1$$

as these two numbers are odd and their difference is 6 and they are not multiples of 6

$$\gcd\Big(f(u_i'), f(u_{i+1})\Big) = \gcd(4i + 1, 4(i + 1) - 1) = \gcd(4i + 1, 4i + 3) = 1$$

as these two number are consecutive odd integers

For $2 \leq i \leq n - 1$ and $i \equiv 1(mod\ 3)$ then $i + 1 \not\equiv 1(mod\ 3)$ and $i - 1 \not\equiv 1(mod\ 3)$

$$\gcd\big(f(u_i), f(v_i)\big) = \gcd(4i + 1, 4i - 2) = 1$$

as one of these numbers is odd and other is even and their difference is 3 and they are not multiple of 3.

$$\gcd\big(f(u_i'), f(v_i')\big) = \gcd(4i - 1, 4i) = 1$$

as these two number are consecutive integers

$$\gcd\big(f(u_i), f(u_{i+1})\big) = \gcd(4i + 1, 4(i + 1) - 1) = \gcd(4i + 1, 4i + 3) = 1$$

as these two number are consecutive odd integers

$$\gcd\big(f(u_{i-1}), f(u_i)\big) = \gcd(4(i - 1) - 1, 4i + 1) = \gcd(4i - 5, 4i + 1) = 1$$

as these two numbers are odd and their difference is 6 and they are not multiples of 3 and 6

$$\gcd\big(f(u_{i-1}), f(u_i')\big) = \gcd(4i - 5, 4i - 1) = 1$$

$$\gcd\big(f(u_i), f(u_{i+1}')\big) = \gcd(4i + 1), 4(i + 1) + 1) = \gcd(4i + 1, 4i + 5) = 1$$

$$\gcd\big(f(u_i'), f(u_{i+1})\big) = \gcd(4i - 1, 4i + 3) = 1$$

as these two numbers are odd and their difference is 4

Thus f is a prime labeling.

Hence G is a prime graph.

Illustration 2.3

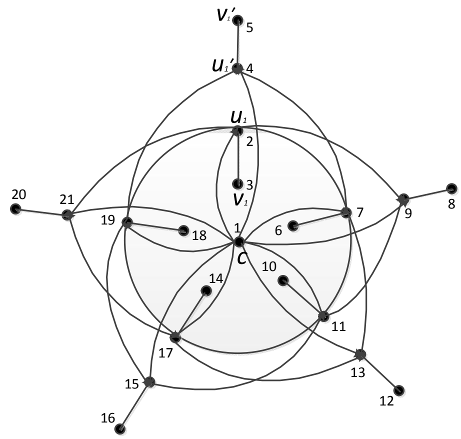

Fig. 3. Prime labeling of duplication of every pendent edge by an edge in Helm H_5

Theorem 2.4. The graph obtained by duplicating every alternate rim edge by an edge in Crown C_n^* is a prime graph if n is even.

Proof: Let $V(C_n^*) = \{ u_i, v_i \ / \ 1 \le i \le n\}$

$$E(C_n^*) = \{u_i u_{i+1} / 1 \le i \le n-1\} \cup \{u_i v_i / 1 \le i \le n\} \cup \{u_n u_1\}$$

Let G be the graph obtained by duplicating every alternate rim edge by an edge in Crown C_n^* and let the new edges be $u_1'u_2', u_3'u_4', \ldots, u_{n-1}'u_n'$ obtained by duplicating the alternate rim edges $u_1u_2, u_3u_4, \ldots, u_{n-1}u_n$ respectively, Then,

$$V(G) = \{ u_i, v_i, u_i' / \ 1 \le i \le n\}$$

$$E(G) = \left\{\frac{u_i u_{i+1}}{1} \le i \le n-1\right\} \cup \{u_i v_i, u_i' v_i / 1 \le i \le n\} \cup \{u_i' u_{i+1}',$$

$$i \text{ is odd}/1 \le i \le n-1\} \cup \{u_i' u_{i+1}, u_i u_{i+1}',$$

$$i \text{ is even}/2 \le i \le n-2\} \cup \{ u_n u_1, u_n' u_1, u_n u_1'\}$$

$$|V(G)| = 3n \ \ and \ \ |E(G)| = 9n/2$$

Define a labeling $f : V(G) \to \{1,2,3, \ldots, 3n\}$ as follows.

Let $f(u_1) = 1, \ f(v_1) = 3n, \ f(u_1') = 3n-1.$

For $2 \le i \le n \ \ and \ i \not\equiv 8 (mod\ 10)$

$$f(u_i) = 3i-4, \quad f(v_i) = 3i-3, \quad f(u_i') = 3i-2$$

For $2 \le i \le n$ and $i \equiv 8 (mod\ 10)$

$$f(u_i) = 3i-2, \ \ f(u_i') = 3i-4,$$

Since $f(u_1) = 1$

$\gcd(f(u_1), f(v_1)) = 1, \gcd(f(u_1), f(u_2)) = 1, \gcd(f(u_1), f(u_n)) = 1,$

$\gcd(f(u_1), f(u_n')) = 1, \gcd(f(u_n), f(u_1')) = \gcd(3n-4, 3n-1) = 1,$

$\gcd(f(u_1'), f(u_2')) = \gcd(3n-1, 4) = 1$

For $2 \le i \le n-1$ and $i \not\equiv 8(mod\ 10)$

$$\gcd\big(f(u_i), f(u_{i+1})\big) = \gcd(3i-4\,, 3(i+1)-4) = \gcd(3i-4\,, 3i-1) = 1$$

among these two numbers one is even and other is odd and their difference is 3 and they are not multiples of 3.

$$\gcd\big(f(u_i), f(v_i)\big) = \gcd(3i-4, 3i-3) = 1$$

$$\gcd\big(f(u_i'), f(v_i)\big) = \gcd(3i-2\,, 3i-3) = 1$$

$\gcd\big(f(u_i'), f(u_{i+1})\big) = \gcd(3i-2,\ 3(i+1)-4) = \gcd(3i-2\,, 3i-1) = 1$, i is even as these two number are consecutive integers

$\gcd\Big(f(u_i), f\big(u_{i+1}'\big)\Big) = \gcd(3i-4\,, 3(i+1)-2) = \gcd(3i-4\,, 3i+1) = 1$, i is even. among these two numbers one is even and other is odd and their difference is 5 and they are not multiples of 5

$\gcd\big(f(u_i'), f(u_{i+1}')\big) = \gcd(3i-2,\ 3(i+1)-2) = \gcd(3i-2\,, 3i+1) = 1$, i is odd among these two numbers one is even and other is odd and their difference is 3 and they are not multiples of 3

For $2 \le i \le n-1$ and $i \equiv 8(mod\ 10)$

$$\gcd\big(f(u_i), f(u_{i+1})\big) = \gcd(3i-2\,, 3i-1) = 1$$

$$\gcd\big(f(u_{i-1}), f(u_i)\big) = \gcd(3(i-1)-4\,, 3i-2) = \gcd(3i-7\,, 3i-2) = 1$$

among these two numbers one is odd and other even and their difference is 5 and they are not multiples of 5

$$\gcd\big(f(u_i), f(v_i)\big) = \gcd(3i-2, 3i-3) = 1$$

$$\gcd\big(f(u_i'), f(v_i)\big) = \gcd(3i-4\,, 3i-3) = 1$$

$\gcd\big(f(u_i'), f(u_{i-1}')\big) = \gcd(3i-4,\ 3(i-2)-2) = \gcd(3i-4\,, 3i-5) = 1$, i is odd as these two numbers are consecutive integers.

$\gcd\big(f(u_i), f(u_{i+1}')\big) = \gcd(3i-2\,, 3(i+1)-2) = \gcd(3i-2\,, 3i+1) = 1$. among these two numbers one is odd and other is even and their difference is 3 and not multiples of 3.

Thus f is a prime labeling.

Hence G is a prime graph.

Illustration 2.4

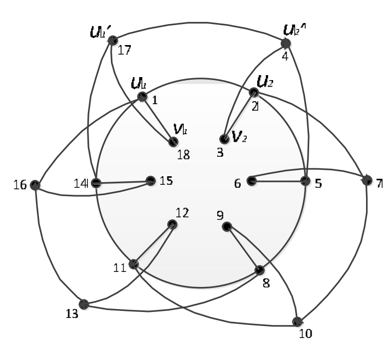

Fig. 4. Prime labeling of duplication of every alternate rim edge by an edge in Crown C_6^*

Theorem 2.5. The graph obtained by duplicating every alternate rim edge by an edge in Helm H_n is a prime graph.

Proof. Let $V(H_n) = \{c, u_i, v_i \ / \ 1 \le i \le n\}$

$$E(H_n) = \{cu_i, u_iv_i \ /1 \le i \le n\} \cup \{u_i u_{i+1} \ / \ 1 \le i \le n-1\} \cup \{u_n u_1\}$$

Let G be the graph obtained by duplicating every alternate rim edge by an edge in Helm H_n and let the new edges be $u_1'u_2', u_3'u_4', \dots, u_{n-1}'u_n'$ obtained by duplicating the alternate rim edges $u_1u_2, u_3u_4, \dots, u_{n-1}u_n$ respectively, Then,

$$V(G) = \{c, u_i, v_i, u_i' / \ 1 \le i \le n\}$$

$$E(G) = \left\{cu_i, cu_i', \frac{u_iv_i, u_i'v_i}{1} \le i \le n\right\} \cup \left\{\frac{u_i \, u_{i+1}}{1} \le i \le n-1\right\} \cup \left\{\frac{u_i'u_{i+1}'}{1} \le i \right.$$

$$\left. \le n-1, \ i \text{ is odd}\right\} \cup \{u_iu_{i+1}', u_i'u_{i+1} \ /1 \le i \le n-2, i \text{ is even}\} \cup \{u_nu_1, u_nu_1', u_n'u_1\}.$$

$$|V(G)| = 3n+1, \qquad |E(G)| = \frac{13n}{2}$$

Define a labeling $f : V(G) \to \{1, 2, 3, \dots, 3n+1\}$ as follows

Let $f(c) = 1$

For $1 \le i \le n$ *and* $i \not\equiv 2 (mod\ 10)$

$$f(u_i) = 3i - 1, \quad f(v_i) = 3i, \quad f(u_i') = 3i + 1$$

For $1 \le i \le n$ and $i \equiv 2 (mod\ 10)$

$$f(u_i) = 3i + 1, \quad f(u_i') = 3i - 1$$

Since $f(c) = 1$

$\gcd(f(c), f(u_i)) = 1$, $\gcd(f(c), f(u_i')) = 1$, $\gcd(f(u_n), f(u_1)) = \gcd(3n-1, 2) = 1$ if n is even

$\gcd(f(u_n), f(u_1')) = \gcd(3n-1, 4) = 1$ if n is even

$\gcd(f(u_n'), f(u_1)) = \gcd(3n+1, 2) = 1$ if n is even

For $1 \le i \le n-1$ *and* $i \not\equiv 2 (mod\ 5)$

$$\gcd(f(u_i), f(u_{i+1})) = \gcd(3i-1, 3(i+1)-1) = \gcd(3i-1, 3i+2) = 1$$

among these two numbers one is odd and other is even and their difference is 3 and they are not multiples of 3

$\gcd(f(u_i), f(v_i)) = \gcd(3i-1, 3i) = 1$

$\gcd(f(u_i'), f(v_i)) = \gcd(3i+1, 3i) = 1$

$\gcd(f(u_i'), f(u_{i+1})) = \gcd(3i+1, 3(i+1)-1) = \gcd(3i+1, 3i+2) = 1$, if i is even

as these two number are consecutive integers

$\gcd(f(u_i'), f(u_{i+1}')) = \gcd(3i+1, 3(i+1)+1) = \gcd(3i+1, 3i+4) = 1$, if i is odd

among these two numbers one is odd and other is even and their difference is 3 and they are not multiples of

3 $\gcd(f(u_i), f(u_{i+1}')) = \gcd(3i-1, 3(i+1)+1) = \gcd(3i-1, 3i+4) = 1$, if i is even.

among these two numbers one is odd and other is even and their difference is 5 and they are not multiples of 5.

For $2 \leq i \leq n-1$ and $i \equiv 2(mod\ 10)$

$$\gcd\big(f(u_i), f(u_{i+1})\big) = \gcd(3i+1\,, 3(i+1)-1) = \gcd(3i+1\,, 3i+2) = 1$$

$$\gcd\big(f(u_i'), f(u_{i+1})\big) = \gcd(3i-1, 3i+2) \quad = 1$$

$$\gcd\big(f(v_i), f(u_i')\big) \quad = \gcd(3i\,, 3i-1) = 1$$

$$\gcd\big(f(u_{i-1}'), f(u_i')\big) = \gcd(3(i-1)+1,\ 3i-1) = \gcd(3i-2\,, 3i-1) = 1$$

as these two numbers are consecutive integers.

$$\gcd\big(f(u_{i-1}), f(u_i)\big) = \gcd(3(i-1)-1\,, 3i+1) = \gcd(3i-4\,, 3i+1) = 1 \ .$$

among these two numbers one is odd and other is even and their difference is 5 and they are not multiples of 5

$$\gcd\big(f(u_i'), f(u_{i+1})\big) = \gcd(3i-1, 3i+2) = 1$$

$$\gcd\big(f(u_i), f(u_{i+1}')\big) = \gcd(3i+1\,, 3(i+1)+1) = \gcd(3i+1\,, 3i+4) = 1$$

among these two numbers one is odd and other is even and their difference is 3 and they are not multiples of 3

Thus f is a prime labeling.

Hence G is a prime graph.

Illustration 2.5

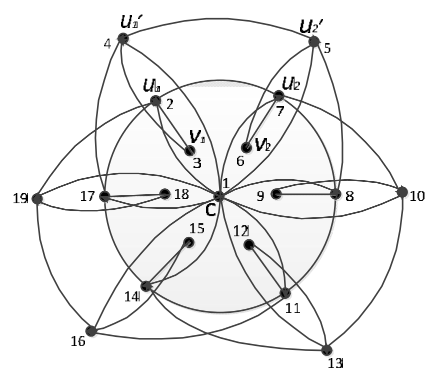

Fig. 5. Prime labeling of duplication of every alternate rim edge by an edge in Helm H_6

Theorem 2.6. The graph obtained by duplicating every alternate rim edge by an edge in Gear graph G_n is a prime graph.

Proof. Let $V(G_n) = \{c, u_i, v_i\,/1 \leq i \leq n\}$

$$E(G_n) = \{cv_i\,, u_iv_i\,/1 \leq i \leq n\} \cup \{u_i\,u_{i+1}\,/\,1 \leq i \leq n-1\} \cup \{u_nu_1\}$$

Let G be the graph obtained by duplicating every alternate rim edge by an edge in Gear graph G_n and let the new edges be $u_1'u_2', u_3'u_4', \dots, u_{n-1}'u_n'$ obtained by duplicating alternate rim edges be $u_1u_2, u_3u_4\,, \dots, u_{n-1}u_n$ respectively, Then

$V(G) = \{c, u_i, v_i, u_i'\ /\ 1 \leq i \leq n\}$

$E(G) = \{cv_i, u_iv_i, u_i'v_i/1 \leq i \leq n\} \cup \{u_i\,u_{i+1}\,/1 \leq i \leq n-1\} \cup \{u_i\,u_{i+1}', u_i'u_{i+1}\,/1$

$\leq i \leq n-2, i$ is even$\} \cup \{u_i' u_{i+1}'/1 \leq i \leq n-1, i$ *is* odd$\} \cup \{u_n u_1, u_n' u_1, u_n u_1'\}$

$|V(G)| = 3n+1,\ |E(G)| = 11n/2$

Define a labeling $f : V(G) \to \{1,2,3,\dots,3n+1\}$ as follows

Let $f(c) = 1$

For $1 \leq i \leq n$ *and* $i \not\equiv 2(mod\ 10)$

$$f(u_i) = 3i-1, \quad f(v_i) = 3i, \quad f(u_i') = 3i+1$$

For $1 \leq i \leq n$ and $i \equiv 2(mod\ 10)$

$$f(u_i) = 3i+1, \quad f(u_i') = 3i-1.$$

Since $f(c) = 1$

$$\gcd(f(c), f(v_i)) = 1, \text{ for } 1 \leq i \leq n$$

Similar to theorem 2.5 we can show that for all other pair of adjacent vertices gcd is 1

Thus f is a prime labeling.

Hence G is a prime graph.

Illustration 2.6

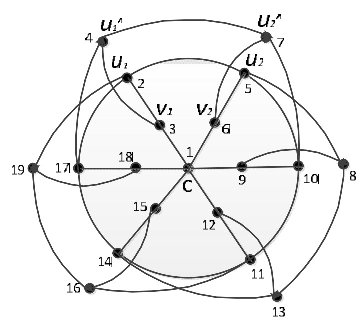

Fig. 6. Prime labeling of duplication of every alternate rim edge by an edge in Gear graph G_6

Theorem 2.7. The graph obtained by duplicating every edge by an edge except the edges incident with central vertex in friendship graph T_n is a prime graph.

Proof. Let $V(T_n) = \{c, u_i \ /\ 1 \leq i \leq 2n+1\}$

$$E(T_n) = \{cu_i\ /1 \leq i \leq 2n\} \cup \{u_i u_{i+1},\ i \text{ is odd } / \ 1 \leq i \leq 2n\}$$

Let G be the graph obtained by duplicating every edge by an edge, except the edges incident with central vertex, in friendship graph T_n, and let the new edges be $u_1'u_2', u_3'u_4', \dots, u_{n-1}'u_n'$ obtained by duplicating the edges $u_1u_2, u_3u_4, \dots, u_{n-1}u_n$ respectively, Then

$V(G) = \{c, u_i, u_i' \ /\ 1 \leq i \leq 2n\}$

$E(G) = \{cu_i, cu_i'/1 \leq i \leq 2n\} \cup \{u_i u_{i+1}, u_i' u_{i+1}'/1 \leq i \leq 2n-1, i$ is odd$\}$.

$|V(G)| = 4n+1,\ |E(G)| = 9n$

Define a labeling $f : V(G) \to \{1, 2, 3, \dots, 4n+1\}$ as follows

For $1 \leq i \leq 2n-1$ and i is odd

$$f(u_i) = 2i, \quad f(u_i') = 2i+2$$

For $1 \leq i \leq 2n$ and i is even

$$f(u_i) = 2i - 1, \quad f(u_i') = 2i + 1$$

Since $f(c) = 1$

$\gcd(f(c), f(u_i)) = 1$, $\gcd(f(c), f(u_i')) = 1$

$\gcd(f(u_i), f(u_{i+1})) = \gcd(2i, 2(i+1) - 1)$

$= \gcd(2i, 2i + 1) = 1$ for $1 \leq i \leq 2n - 1$ and i is odd

$\gcd\left(f(u_i'), f(u_{i+1}')\right) = \gcd(2i + 2, 2(i+1) + 1)$

$= \gcd(2i + 2, 2i + 3) = 1$ for $1 \leq i \leq 2n - 1$ and i is odd

as these two numbers are consecutive integers

Thus f is a prime labeling.

Hence G is a prime graph.

Illustration 2.7

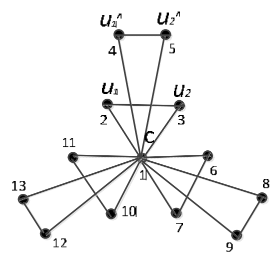

Fig. 7. Prime labeling of duplication of every edge by an edge except the edges incident with central vertex in friendship graph T_3

Conclusion

Seven new families of prime labeling of graphs are investigated. To investigate more edge duplication of prime labeling of graphs and to discuss this labeling in the context of various graph operations is an open area of research.

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