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NIL ELEMENTS AND NONCOMMUTATIVE RINGS

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Nil elements (which are special type of nilpotent elements) have been introduced recently. Here we study the role of nil elements in noncommutative rings and provide some results. Among other things we prove that a noncommutative ring of order n can contain maximum $n-1$ nil elements and it is noted that if R be a noncommutative ring, N and N' are the sets of all nil and nilpotent elements of R respectively then N is a nil ideal of R iff N is a subring of R and $(x+ax) \in N, (x+xa) \in N$, $xax = 0, \forall x \in N, a \in R$ provided R has non-nil nilpotent elements.

KEYWORDS : Even square rings, nil elements, nilpotent elements, noncommutative rings, nil ideals.

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Introduction

Nil elements do not appear explicitly in the mathematical literatures [1-3] and the notion of nil elements has been introduced in [4]. It has been seen that the set of all nil elements in a commutative ring forms a nil ideal of the ring however the same is not necessarily true in the case of noncommutative rings. Recall that an element a of a ring R is called a nil element if $a^2 = 2a = 0$ and a ring R is called an even square ring if $a^2 \in 2R, \forall a \in R$.

We notice that if R is a noncommutative ring containing nil elements then the sum and product of any two non-nil nilpotent elements are not necessarily non-nil nilpotent elements. We prove that a noncommutative ring of order n can contain maximum $n-1$ nil elements. We determine certain conditions under which the set N of all nil elements in a non-commutative ring forms a nil ideal of the ring.

We also prove that if R be a noncommutative ring and N be the set of all nil elements of R then the following are equivalent

1. N is an ideal of R
2. N is a subring of R and $x + ax \in N, \ \forall x \in N, a \in R$
3. N is a left ideal of R.

provided R does not have any non-nil nilpotent element.

Some results

Proposition 1. Let R be a finite noncommutative ring containing nil elements then the sum and product of non-nil nilpotent elements are not necessarily non-nil nilpotent elements.

Proof. See example 1.

Proposition 2. A noncommutative ring of order n can contain maximum $n-1$ nil elements.

Proof. Any ring in which every element is nil is a commutative even square ring [4]. Therefore each element of a noncommutative ring cannot be nil. Now it is sufficient to prove that there exists a noncommutative ring of order n containing $n-1$ nil elements. We shall construct such a ring. Let $R = \{0, a, b, ab, a+b, a+ab, b+ab, a+b+ab\}$. Here a and b are any two non-zero distinct nil elements of a noncommutative ring R' such that $ba = 0$. It is easy to see that R is a noncommutative ring of order 8 under addition and multiplication defined in R' and R contains 7 nil elements. Thus a noncommutative ring of order n can contain maximum $n-1$ elements.

Proposition 3. In a noncommutative ring the set of all nil elements does not necessarily form an ideal of the ring.

Proof. Refer the ring R given in the proof of proposition 3.

Remark 2. The set of all nil elements in a commutative ring forms an ideal of the ring [4].

Proposition 4. Let R be a noncommutative ring and N be the set of all nil elements of R then N is a subring of R provided $x+y \in N, \forall x, y \in N$.

Proof. Let R be a noncommutative ring and N be the set of all nil elements of R such that $x+y \in N, \forall x, y \in N$. We have to prove that $xy \in N, \forall x, y \in N$. $(x+y)^2 = 0 \Rightarrow (xy)^2 = 0$. Clearly, $2xy = 0$. Therefore, $xy \in N, \forall x, y \in N$. Hence N is a subring of R.

Proposition 5. Let R be a noncommutative ring and N be the set of all nil elements of R then N is an ideal of R iff

1. N is a subring of R and
2. $x+ax$ and $x+xa \in N, xax = 0, \forall x \in N, a \in R$.

Corollary 1. Let R be a noncommutative ring and N be the set of all nil elements of R then $x+xa$ is a nilpotent element of R for each $x \in N$, $a \in R$ provided $xax = 0, \forall x \in N, a \in R$.

Proposition 6. Let R be a noncommutative ring and N be the set of all nil elements of R then the following are equivalent

1. N is an ideal of R
2. N is a subring of R and $x+ax \in N$, $\forall x \in N, a \in R$
3. N is a left ideal of R

provided R does not have any non-nil nilpotent element that is $N = N'$.

Proof. We prove that $3 \Rightarrow 1$ (proof for remaining parts is easy) if N is a left ideal of R then $ax \in N, \forall a \in R, x \in N$. $ax \in N \Rightarrow (ax)^2 = 0$. Therefore $(xa)^3 = 0$. Since R does not have any non-nil nilpotent elements so $(xa)^3 = 0 \Rightarrow (xa)^2 = 0$. Thus $xa \in N, \forall a \in R, x \in N$. Hence N is an ideal of R.

Proposition 7. Let R be a noncommutative ring and N be the set of all nil elements of R then the following are equivalent

1. N is an ideal of R
2. N is a subring of R and $x + xa \in N,\ \forall x \in N, a \in R$
3. N is a right ideal of R

provided R does not have any non-nil nilpotent element that is $N = N'$. Here N' denotes the set of all nilpotent elements of R.

Proof. Same as above.

Corollary 2. Let R be a noncommutative ring and N be the set of all nil elements of R then $x + xa$ is a nilpotent element of R for each $x \in N, a \in R$ if $N = N'$. Here N' denotes the set of all nilpotent elements of R.

Some examples

Example 1.

$$R = \left\{ \begin{matrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 9 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 6 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 9 & 9 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 6 & 3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 9 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 9 & 3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 9 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 9 & 6 \\ 0 & 0 \end{pmatrix} \end{matrix} \right\}$$

Clearly R is a noncommutative ring under matrix addition and multiplication modulo 12.

Let N and N' denote the set of all nil and nilpotent elements of R respectively. Then

$$N = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 6 \\ 0 & 0 \end{pmatrix} \right\}$$

and
$$N' = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 6 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 9 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 9 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 3 \\ 0 & 0 \end{pmatrix} \right\}.$$

One may easily verify that the sum and product of all non-nil nilpotent elements in R are nil elements.

Example 2. Ring R (given above) provides an example of a ring in which N and N' both form nil ideal of R.

Example 3. Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ then R is a non commutative ring of order 4 under addition and multiplication of matrices modulo 2. Let

$N = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$. Clearly R does not have any non-nil nilpotent element. N is the only non-zero nil ideal of R whose every element is nil.

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