Acta Ciencia Indica, Vol. XLIII M, No. 1, 55 (2017)

A STUDY OF CERTAIN NEW CURVES IN AN EUCLIDEAN SPACE OF THREE DIMENSIONS-II

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RECEIVED : 24 September, 2015

Analogous to a curve, called λ_0-curve, which is such that is osculating plane contains the vector $d^3\lambda^i/ds^3$ Rastogi [5], in this paper, we have defined a new curve called λ_N-curve. This is such that its normal plane contains the vector $d^3\lambda^i/ds^3$, where λ^i are the contra-variant components of a unit vector in the direction of the line 1 of the congruence passing through a point *P*. In this paper we have studied some of its curvature properties in an Euclidean space of three dimensions via-a-vis other well known curves.

PRELIMINARIES

Let S: $x^i = x^i\ (u^\alpha)$, (i = 1, 2, 3 and α = 1, 2), be the surface of reference of rectilinear congruence, a line 1 of which is given by the direction cosines

$$\lambda^i = \lambda^i\ (u^\alpha),\ \lambda^i . \lambda^i = 1 \qquad \dots (1.1)$$

We assume that x^i and λ^i are continuous along with their partial derivatives up to the required order. At any point $P\ (x^i)$ of S, λ^i is expressible as [3]

$$\lambda^i = p^\alpha x^i,_\alpha + q X^i, \qquad \dots (1.2)$$

where p^α are the contra-variant components of a vector in S at P and q is a scalar function, X^i are the direction cosines of the normal to S at P and $x^i,_\alpha$ denotes the covariant derivative of x^i with respect to u^α based on the fundamental tensor of S, $g_{\alpha\beta} = x^i,_\alpha . x^i,_\beta$.

The Gauss and Weingarten equations in Eisenhart [2] are given by $x^i,_{\alpha\beta} = d_{\alpha\beta} X^i$, $X^i,_\alpha = - d_\alpha{}^\delta x^i,_\delta$, where $d_{\alpha\beta}$ is the second fundamental tensor of the surface S.

Let us consider a curve $C : x^i = x^i\ (s)$ on S, then the intrinsic derivative of x^i, $d\,x^i/d\,s$ and $d^2 x^i/d\,s^2$ is expressed as

$$x'^i = d\,x^i/ds = x^i,_\alpha u'^\alpha,\ x''^i = d^2x^i/ds^2 = \rho^\alpha x^i,_\alpha + X^i k_n, \qquad \dots (1.3)$$

and $$x'''^i = d^3 x^i/ds^3 = (\rho^\alpha,_\beta - k_n\ d_{\beta\theta}\, g^{\theta\alpha})\, x^i,_\alpha u'^\beta + (k_{n,\beta} + \rho^\alpha d_{\alpha\beta})\, X^i u'^\beta, \qquad \dots (1.4)$$

where, primes indicate the differentiation with respect to are-length s, p^α are the components of the geodesic curvature vector of the curve C and k_n is the normal curvature of the surface in the direction of the curve C [2].

Similar to above equations we can also obtain for a vector λ^i in the direction of the curves of the congruence-λ, following intrinsic derivatives Rastogi and Bajpai [4].

$$\lambda' = d\,\lambda^i/ds = \lambda^i, \alpha\, u'\ \alpha = (\mu^\gamma{}_\alpha\, x^i,_\gamma + \nu_\alpha X^i)\, u'^\alpha, \qquad \dots (1.5)$$

where $$\mu^\gamma{}_\alpha = p^\gamma,_\alpha - q\, d_{\alpha\beta}\, g^{\beta\gamma},\ \nu_\alpha = q,_\alpha + p^\beta\ d_{\alpha\beta}. \qquad \dots (1.6)$$

For a normal congruence $\mu^{\gamma}_{\alpha} = - d^{\gamma}_{\alpha}$ and $v_{\alpha} = 0$, while for a congruence formed by tangents to a one parameter family of curves $\mu^{\gamma}_{\alpha} = p^{\gamma},_{\alpha}$ and $\nu_{\alpha} = p^{\beta} d_{\alpha\beta}$. We known that $\lambda^i . \lambda^i = 1$, therefore we can obtain $\lambda^i . \lambda^i,_{\alpha} = 0$, which gives $p_{\gamma} \mu^{\gamma}_{\alpha} + q\, v_{\alpha} = 0$.

Differentiating $\lambda^i,_{\alpha}$ covariantly with respect to u'^{β}, we get

$$\lambda^i,_{\alpha\beta} = M^{\gamma}{}_{\alpha\beta}\, x^i, \gamma + N_{\alpha\beta} X^i, \qquad \dots (1.7)$$

where $$M^{\gamma}{}_{\alpha\beta} = \mu^{\gamma}{}_{\alpha,\beta} - \nu_{\alpha} d_{\beta\theta}\, g^{\theta\gamma}, N_{\alpha\beta} = \nu_{\alpha,\beta} + \mu^{\gamma}_{\alpha} d_{\gamma\beta}. \qquad \dots (1.8)$$

The intrinsic derivative of $d\lambda^i/ds$, represented by λ''^i along C can be obtained as follows:

$$\lambda''^i = (M^{\gamma}{}_{\alpha\beta}\, \mu'^{\alpha} u'^{\beta} + \mu^{\gamma}_{\alpha} u''^{\alpha})\, x^i,_{\gamma} + (N_{\alpha\beta} u'^{\alpha} u'^{\beta} + \nu_{\alpha} u''^{\alpha}) X^i, \qquad \dots (1.9)$$

such that $p^{\gamma} M^{\gamma}_{\alpha\beta} + q\, N_{\alpha\beta} + \mu_{\alpha\delta}\, \mu^{\delta}_{\beta} + \nu_{\alpha} \nu_{\beta} = 0$.

From equation (1.7) we can get

$$\lambda^i,_{\alpha\beta\gamma} = (M^{\theta}{}_{\alpha\beta},_{\gamma} - N_{\alpha\beta}\, d_{\gamma\delta}\, g^{\delta\theta})\, x^i,_{\theta} + X^i (M^{\theta}{}_{\alpha\beta}\, d_{\theta\gamma} + N_{\alpha\beta,\gamma}) \qquad \dots (1.10)$$

such that $$q\, \{M^{\theta}{}_{\alpha\beta}\, (q\, d_{\theta\gamma} + \mu_{\theta\gamma} - p_{\theta,\gamma}) + M^{\theta}{}_{\beta\gamma}\, \mu_{\theta\alpha} - M_{\beta\theta,\gamma}\, \mu^{\theta}{}_{\alpha})$$
$$- \mu^{\theta}{}_{\alpha}\, \mu^{\varphi}{}_{\beta}\, d_{\theta\gamma}\, p_{\varphi} - N_{\alpha\beta}\, (\mu^{\theta}{}_{\gamma}\, p_{\theta} + q\, p^{\theta} d_{\theta\gamma} + q\, q,_{\gamma}) = 0. \qquad \dots (1.11)$$

The intrinsic derivative of $d^2\lambda^i/ds^2$ along C which is represented by λ'''^i can be obtained in the following form

$$\lambda'''^i = (x^i,_{\gamma} A^{\gamma} + BX^i), \qquad \dots (1.12)$$

where

$$A_{\gamma} = [\mu^{\gamma}_{\alpha}\, u'''^{\alpha} + u''^{\alpha}\, u'^{\beta}\, (2M^{\gamma}{}_{\alpha\beta} + M^{\gamma}{}_{\beta\alpha}) + u'^{\alpha}\, u'^{\beta}\, u'^{\delta}\, (M^{\gamma}{}_{\alpha\beta},_{\delta} - N_{\alpha\beta}\, d^{\gamma}{}_{\delta})] \qquad \dots (1.13)\text{ a}$$

and

$$B = [\nu_{\alpha}\, u'''^{\alpha} + u''^{\alpha} + u''^{\alpha}\, u'^{\beta}\, (2N_{\alpha\beta} + N_{\beta\alpha}) + u'^{\alpha}\, u'^{\beta}\, u'^{\delta}\, (d_{\gamma\delta}\, M^{\gamma}{}_{\alpha\beta} + N_{\alpha\beta},_{\delta})]. \qquad \dots (1.13)\text{ b}$$

λ_N – CURVES

Definition 2.1 – A curve C on the surface of reference S shall be called λ_N –curve in an Euclidean space of three dimensions if the normal plane at any point P of C contains the vector λ'''^i.

Education of a normal plane to curve C at a point P is given by Eisenhart [2] as

$$('x^i - x^i)\, x'^i = 0 \qquad \dots (2.1)$$

For C to be a λ_N-curve, $'x^i = x^i + t\, \lambda'''^i$, must satisfy equation (2.1). Hence in view of (1.12) and (2.1), we obtain $A_{\lambda}\, u'^{\gamma} = 0$ or alternatively

$$[\mu_{\gamma\alpha}\, u'''^{\alpha} + u''^{\alpha}\, u'^{\beta}\, (2M_{\gamma\alpha\beta} + M_{\gamma\beta\alpha}) + u'^{\alpha}\, u'^{\beta}\, u'^{\delta}\, (M_{\gamma\alpha\beta},_{\delta} - N_{\alpha\beta}\, d_{\gamma\delta})]\, u'^{\gamma} = 0, \qquad \dots (2.2)$$

as the differential equation of a λ_N–curve.

Thus we have

Theorem 2.1 – In an Euclidean space of three dimensions, the differential equations of a λ_N-curve is given by either $A_{\gamma}\, u'^{\gamma} = 0$ or equation (2.2).

For a normal congruence equation (2.2) reduces to

$$u'^{\gamma}\, \{d_{\gamma\alpha}\, u'''^{\alpha} + u''^{\alpha}\, u'^{\beta}\, (2d_{\gamma\alpha,\beta} + d_{\gamma\beta,\alpha}) + u'^{\alpha}\, u'^{\beta}\, u'^{\delta}\, d_{\gamma\alpha\beta,\delta}\}$$
$$- K_n\, d^{\gamma}{}_{\alpha}\, d_{\gamma\beta})\, u'^{\alpha}\, u'^{\beta} = 0, \qquad \dots (2.3)$$

While for a congruence formed by tangents to a one parameter family of curves, equation (2.2) reduces to

$$u'^{\alpha}\,\{p_{\alpha,\beta}\,u'''^{\beta} + u''^{\gamma}\,u'^{\beta}\,(2p_{\alpha,\gamma\beta} - 2p_{\alpha,\beta\gamma} - p^{\varphi}\,d_{\gamma\varphi}\,d_{\alpha\beta})]$$

$$+\,[(p_{\alpha,\theta,\beta,\delta} - p^{\varphi}\,d_{\theta\varphi}\,d_{\alpha,\beta,\delta})\,u'^{\alpha}\,u'^{\beta} - k_n\,(3p^{\varphi},{}_{\delta}d_{\theta\varphi} + 2p^{\varphi}\,d_{\theta\varphi,\delta})]\,u'^{\delta}\,u'^{\theta} = 0. \quad \ldots (2.4)$$

ALTERNATIVE EQUATION OF λ_N-CURVES

Let α^i, β^i and γ^i be respectively the direction cosines of unit tangent, principal normal and bi-normal to a curve *C*, then for a λ_N-curve, λ'''^i can be expressed as follows:

$$\lambda'''^i = a\,\beta^i + b\,\gamma^i, \quad \ldots (3.1)$$

where *a* and *b* are arbitrary constants to be determined.

Let ψ be the angle between the vectors β^i and λ'''^i, then for $D\,|\lambda'''^i| = 1$,

$$a = D^{-1}\cos\psi, \quad b = \pm D^{-1}\sin\psi. \quad \ldots (3.2)$$

Substituting in equation (3.1) from (1.3), (1.12) and using

$$\gamma^i = -\,\tau^{-1}\,(k\,\alpha^i + d\,\beta^i/ds), \quad \ldots (3.3)$$

We obtain on simplification

$$\eta^{\alpha} = A^{\alpha} + D^{-1}\,[\tau^{-1}\sin\psi\,u'^{\beta}\,\{k\,\delta^{\alpha}_{\beta} + k^{-1}\,(\rho^{\alpha},{}_{\beta} - k_n\,d^{\alpha}_{\beta})\}$$

$$-\,k^{-1}\,\rho^{\alpha}\,(\cos\psi + k^{-1}\,\tau^{-1}\,k'\sin\psi)] = 0 \quad \ldots (3.4)$$

And $$\tau = \sin\psi\,(d_{\alpha\beta}\,\rho^{\alpha}\,u'^{\beta} - k_n\,k'\,k^{-1} + k_n')\,(k_n\cos\psi - B\,k\,D)^{-1} \quad \ldots (3.5)$$

Equation (3.4) represented the differential equation of a λ_N-curve and the vector η^{α} is called λ_N-curvature vector of a curve *C* and η^{α} vanishes for a λ_N-curve.

The differential equation of a Darboux curve Semin [6] is expressed as $d_{\alpha\beta}\rho^{\alpha}u'^{\beta} + kn' = 0$, therefore from equation (3.5), for a Darboux curve we can obtain

$$\tau = -\sin\psi\,(k_n\,k'\,k^{-1})\,(k_n\cos\psi - B\,k\,D)^{-1}, \quad \ldots (3.6)$$

which implies.

Theorem 3.1. In an Euclidean space of three dimensions the torsion of a λ_N-curve, which is also a Darboux curve is given by (3.6)

Now we shall discuss some special cases.

Case I. If the vector λ'''^i is parallel to vector β^i, with the help of equation (3.2), (3.4) and (3.5) we get $\tau = 0$ and

$$\eta^{\alpha} = A^{\alpha} - D^{-1}\,k^{-1}\,\rho^{\alpha}. \quad \ldots (3.7)$$

Hence we have:

Theorem 3.2. In an Euclidean space of three dimensions, if the vector λ'''^i is parallel to the vector β^i, the vector A^{α} is parallel to the vector ρ^{α} and the λ_N-curvature vector η_{α} satisfies $\eta_{\alpha}\,u'^{\alpha} = 0$ such that the torsion of the curve C vanishes identically.

Case II. If the vector λ'''^i is perpendicular to vector β^i, with the help of equation (3.2), (3.4) and (3.5) we get

$$A^{\alpha} = D^{-1}\,k^{-1}\,\tau^{-1}\,[k^{-1}\,k'\,\rho^{\alpha} + k_n\,d^{\alpha}_{\beta}\,u'^{\beta} - k^2\,u'^{\alpha} - \rho^{\alpha},{}_{\beta}\,u'^{\beta}] \quad \ldots (3.8)$$

and $$\tau = (B\,k\,D)^{-1}\,(k_n\,k'\,k^{-1} - k_n' - d_{\alpha\beta}\,\rho^{\alpha}\,u'^{\beta}) \qquad \ldots (3.9)$$

From equation (3.8), by virtue of $\rho_{\alpha}\,u'^{\alpha} = 0$, $\rho_{\alpha,\,\beta}\,u'^{\alpha}\,u'^{\beta} = 0$ and $k^2 = k_n^2 + k_g^2$, we get on simplification

$$D^{-1}\,k^{-1}\,\tau^{-1}\,k_g^{2} = 0, \qquad \ldots (3.10)$$

which leads to $k_g = 0$ or $k_n = k$, provided D, k and τ do not vanish.

Hence we have:

Theorem 3.3. In an Euclidean space of three dimensions, if the vector λ'''^{i} is perpendicular to the vector β^i such that D, k and τ do not vanish then the first curvature of the λ_N-curve is equal to the normal curvature.

From equation (3.9), for a Darboux curve, we can obtain $\tau = (B\,k\,D)^{-1}\,k_n\,k'\,k^{-1}$, which leads to

Theorem 3.4. In an Euclidean space of three dimensions, if the vector λ'''^{i} is perpendicular to the vector β^{i} and the given curve C is both a λ_N-curve and a Darboux curve, then its torsion is given by $\tau = (B\,k\,D)^{-1}\,k_n\,k'\,k^{-1}$.

λ_N-CURVATURE.

Analogous to the well known definition of union curvature, let K_N be the λ_N-curvature of a curve C, *i.e.*, the magnitude of the vector η^{α}, then we can obtain after some calculation

$$\begin{aligned}
K^2_N = {} & A^{\alpha} A_{\alpha} + 2D^{-1}\,[\tau^{-1}\sin\psi\{((k-k_n)\,d_{\alpha\beta} + k^{-1}\rho_{\alpha,\,\beta})\,A^{\alpha}\,u'^{\beta} - k^{-2}\,k'\,\rho^{\alpha}\,A_{\alpha}\} \\
& - k^{-1}\,\rho^{\alpha}\,A_{\alpha}\cos\psi] + D^{-2}\,[\tau^{-2}\sin^2\psi\,\{k^2 - 2k_n^2 + k^{-4}\,k'^2\,k_g^2 \\
& + (k^{-2}\,\rho_{\alpha,\,\delta}\,\rho^{\alpha}{}_{,\,\beta} - 2k^{-2}\,k_n\,\rho^{\alpha}{}_{,\,\beta}\,d_{\alpha\delta} + k^2\,k_n^2\,d_{\alpha\,\delta}\,d^{\alpha}{}_{\beta} + 2\rho_{\delta,\,\beta})\,u'^{\beta}\,u'^{\delta}\} \\
& + 2k^{-2}\,\tau^{-1}\sin\psi\,\{\cos\psi\,(k^{-1}\,k'\,k_g^2 - \rho^{\alpha}{}_{,\,\beta}\,\rho_{\alpha}u'^{\beta} + k_n\,\rho^{\alpha}\,d_{\alpha\beta}\,u'^{\beta}) \\
& - k^{-1}\,\tau^{-1}\,k'\,\rho^{\alpha}\,u'^{\beta}\sin\psi\,(p^{\alpha}{}_{,\,\beta} - k_n\,d^{\alpha}{}_{\beta})\} + k^{-2}\,k_g^2\cos^2\psi]. \qquad \ldots (4.1)
\end{aligned}$$

From equation (4.1) we obtain

Theorem 4.1. In a three- dimensional Euclidean space λ_N- curvature K_N, of a curve C is given by equation (4.1) and it vanishes for a λ_N-curve.

CURVATURE OF A λ_N-CURVE,

Let us assume that

$$\beta^i = c\,\gamma^i + d\,\lambda'''^{i} \qquad \ldots (5.1)$$

If θ is the angle between γ^i and λ'''^{i}, we can easily obtain

$$c = -\cot\theta, \quad d = D\,\mathrm{cosec}\,\theta. \qquad \ldots (5.2)$$

Substituting the values of c and d in (5.1) together with the value of β^i, γ^i and λ'''^{i}, we get on simplification

$$\begin{aligned}
A^{\alpha} = {} & (D\,k\,\mathrm{cosec}\,\theta)^{-1}\,\{\rho^{\alpha} - \tau^{-1}\cos\theta\,(k\,u'^{\alpha} + k^{-1}\,\rho^{\alpha}{}_{,\,\beta}\,u'^{\beta}) \\
& - d^{\alpha}{}_{\beta}\,u'^{\beta} - k^{-2}\,k'\,\rho^{\alpha})\} \qquad \ldots (5.3)
\end{aligned}$$

and $$k_n = B\,k\,D\,\mathrm{cosec}\,\theta + k^{-1}\,\tau^{-1}\cos\theta\,(k_n' + d_{\alpha\beta}\,\rho^{\alpha}\,u'^{\beta} - k^{-1}\,k_n\,k'). \qquad \ldots (5.4)$$

Multiplying equation (5.3) by u'^{α} and using $u'^{\alpha}\,\rho_{\alpha} = 0$, $\rho_{\alpha,\,\beta}\,u'^{\beta}\,u'^{\alpha} = 0$, $A_{\alpha}\,u'^{\alpha} = 0$, we get on simplification $k = k_n$ or $k_g = 0$. Substituting $k = k_n$ in (5.4), we get

$$k^2 = (\tau\, DB)^{-1} \cos\theta\, d_{\alpha\beta}\, \rho_{\alpha}\, u'^{\beta} (\sin\theta - DB)^{-1}. \quad \ldots (5.5)$$

If the given curve is a line of curvature Eisenhart [2], *i.e.*, $d_{\alpha\beta}\,\rho^{\alpha}\,u'^{\beta} = 0$, equation (5.5) gives $k = 0$. Hence we have:

Theorem 5.1. In an Euclidean space of three dimensions, if λ_N-curve is also lines of curvature its curvature k vanishes.

Some properties of other curves

Case I. If we are given a curve which satisfies the differential equation $\lambda^i\, x'^i = 0$ or alternatively $p_{\alpha}\,u'^{\alpha} = 0$, *i.e.*, if it is a C_B-cure Bhattacharya [1], we can obtain on differentiating this equation $\lambda'^i\, x'^i + \lambda^i\, x''^i = 0$ or alternatively

$$\mu_{\alpha\beta}\, u'^{\alpha}\, u'^{\beta} + (p_{\alpha}\,\rho^{\alpha} + q\, k_n) = 0. \quad \ldots (6.1)$$

If we assume that the given curve is also a hyper-asymptotic curve [3], *i.e.*, $p_{\alpha}\rho^{\alpha} + qk_n = 0$, by virtue of equation (6.1) we get $\mu_{\alpha\beta}\, u'^{\alpha}\, u'^{\beta} = 0$, which shows that the given curve is an N^*-curve Trivedi [6]. Hence we have:

Theorem 6.1. In a three dimensional Euclidean space the necessary and sufficient condition for the C_B-curves to be a hyper-asymptotic curve is that it be an N^*- curve.

Case II. If we assume that $\lambda''^i\, X^i = 0$, *i.e.*, $N_{\alpha\beta}\, u'^{\alpha}\, u'^{\beta} = 0$, we can obtain

$$\lambda'''^i\, X^i - \lambda''^i\, d^{\delta}_{\alpha}\, u'^{\alpha}\, x^i_{,\,\delta} = 0, \quad \ldots (6.2)$$

which can be expressed as $B = M_{\delta}\, d^{\delta}_{\alpha}\, u'^{\alpha}$. Hence we have:

Theorem 6.2.– In a three- dimensional Euclidean space, a curve C, satisfying $N_{\alpha\beta}\, u'^{\alpha}\, u'^{\beta} = 0$, also satisfies $B = M_{\delta}\, d^{\delta}_{\alpha}\, u'^{\alpha}$.

Case III. If for a curve C in an Euclidean space of three dimensions $\lambda''^i\, x'^i = 0$, we get on differentiation $\lambda'''^i\, x'^i + \lambda'''^i\, x''^i = 0$, which leads to

$$A_{\alpha}\, u'^{\alpha} + M_{\alpha}\,\rho^{\alpha} + N\, k_n = 0. \quad \ldots (6.3)$$

If $A_{\alpha}\, u'^{\alpha} = 0$, the given curve is a λ_N-curve, while if $M_{\alpha}\,\rho^{\alpha} + N\, k_n = 0$, the given curve is a generalized asymptotic curve [4]. Hence we have:

Theorem 6.3. In a three-dimensional Euclidean space a curve C satisfying $\lambda''^i\, x'^i = 0$ or alternatively $M_{\alpha}\, u'^{\alpha} = 0$, is a generalized asymptotic curve if and only if it is a λ_N-curve.

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