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A STUDY OF CERTAIN NEW CURVES IN AN EUCLIDEAN SPACE OF THREE DIMENSIONS–I

BAJPAI PRADEEP, RASTOGI S.C.

Department of Mathematics, Bhabha Institute of Technology, Kanpur (D) U.P. (India)

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In this paper I have defined a curve which is such that its osculating plane contains the vectors $\frac{d^3\lambda^i}{ds^3}$, where λ^i are the contra–variant components of a unit vector in the direction of the line ℓ of the congruence passing through a point *P*. We have called such a curve as λ_0–curve in an Euclidean space of three dimensions and studied some of its curvature properties.

INTRODUCTION

Let $S : = x^i = x^i\,(u^\alpha)$, $(i = 1, 2, 3$ and $\alpha = 1, 2)$, be the surface of reference of a rectilinear congruence, a line ℓ of which is given by the direction cosines

$$\lambda^i = \lambda^i\,(u\alpha),\quad \lambda^i \cdot \lambda^i = 1 \qquad \ldots (1.1)$$

We assume that x^i and λ^i are continuous along with their partial derivatives up to the required oreder. At any point $P\,(x^i)$ of S, λ^i is expressible as [3]

$$\lambda^i = p^\alpha\, x^i_a + q\, X^i \qquad \ldots (1.2)$$

where p^α are the contra-variant components of a vector in S at P and q is a scalar function, X^i are the direction cosines of the normal to S at P and x^i_α denotes the covariant derivatives of x^i with respect to u^α based on the fundamental tensor of S, $g_{\alpha\beta} = x^i_\alpha \cdot x^i_\beta$.

The Gauss and Weingarten equations are given by Eisenhart [1] as follows:

$$x^i_{\alpha\beta} = d_{\alpha\beta}\, X^i,\ X^i_\alpha = -\, d_{\alpha\beta}\, g_{\beta\delta}\, x^i_\delta \qquad \ldots (1.3)$$

where $d_{\alpha\beta}$ is the second fundamental tensor of the surface S.

Let us consider a curve $C : x^i = x^i\,(s)$ on S, then the intrinsic derivatives of x^i, $\frac{dx^i}{ds}$ and $\frac{d^2x^i}{ds^2}$ are expressed as

$$x'^i = \frac{dx^i}{ds} = x^i_{,\alpha}\, u'^\alpha,\ x''^i = \frac{d^2x^i}{ds^2} = \rho^\alpha x^i_{,\,\alpha} + X^i\, k_n, \qquad \ldots (1.4)$$

and

$$x'''^i = (\rho^\alpha_{,\,\beta} - k_n\, d_{\beta\,\theta}\, g^{\theta\alpha})\, x^i_{,\,\alpha}\, u'^\beta + (k_{n,\,\beta} + \rho^\alpha\, d_{\alpha\beta})\, X^i\, u'^\beta \qquad \ldots (1.5)$$

where primes indicate the differentiation with respect to arc–length s, p^α are the components of the geodesic curvature vector of the curve C in two dimensional Euclidean space and k_α is the normal curvature of the surface in the direction of the curve C [1]. For a normal congruence equation (1.2) gives $p^{\alpha} = 0$ and $q = 1$, while for a congruence formed of tangents to a one parameter family of curves $q = 0$ and p^{α} is a unit vector in a two dimensional Euclidean space.

Intrinsic derivatives of vectors λ^i

Similar to equations (1.4) and (1.5) we can also obtain for a vector λ^i in the direction of the curves of the congruence-λ, following;

Intrinsic derivatives

$$\lambda'^i = \frac{d\lambda^i}{ds} = \lambda^i{}_{,\alpha}\, u'^{\alpha} = (\mu^{\gamma}{}_{\alpha}\, x^i{}_{,\gamma} + v_{\alpha} X^i)\, u'^{\alpha}, \quad \ldots (2.1)$$

where $\mu^{\gamma}{}_{\alpha} = p^{\gamma}{}_{,\alpha} - q\, d_{\alpha\beta}\, g^{\beta\gamma},\ v_{\alpha} = q_{,\alpha} + p^{\beta}\, d_{\alpha\beta}$ … (2.2)

Differentiating $\lambda^i{}_{,\alpha}$ covariantly with respect to u'^{β}, we get

$$\lambda^i{}_{,\alpha\beta} = M^{\gamma}{}_{\alpha\beta}\, x^i{}_{,\gamma} + N_{\alpha\beta}\, X^i, \quad \ldots (2.3)$$

and $M^{\gamma}{}_{\alpha\beta} = \mu^{\gamma}{}_{\alpha,\beta} - v_{\alpha}\, d_{\beta\theta}\, g^{\theta\gamma},\ N_{\alpha\beta} = v_{\alpha,\beta} + \mu^{\gamma}{}_{\alpha}\, d_{\gamma\beta}$ … (2.4)

Since we know that λ^i is unit vector, therefore we can obtain $\lambda^i \cdot \lambda^i{}_{,\alpha} = 0$, which gives $p_{\gamma}\, \mu^{\gamma}{}_{\alpha} + q\, v_{\alpha} = 0$. The intrinsic derivative of $\frac{d\lambda^i}{ds}$, represented by λ''^i along C can be obtained as follows:

$$\lambda''^i = (M^{\gamma}{}_{\alpha\beta}\, u'^{\alpha}\, u'^{\beta} + \mu^{\gamma}{}_{\alpha}\, u''^{\alpha})\, x^i{}_{,y} + (N_{\alpha\beta}\, u'^{\alpha}\, u'^{\beta} + v_{\alpha}\, u''^{\alpha}) X^i \quad (2.5)$$

where $p_{\gamma}\, M^{\gamma}{}_{\alpha\beta} + q\, N_{\alpha\beta} + \mu^{\alpha}{}_{\delta}\, \mu^{\delta}{}_{\beta} + v_{\alpha}\, v_{\beta} = 0$, which is a consequence of $p_{\gamma}\mu^{\gamma}{}_{\alpha} + qv_{\alpha} = 0$. From equation (2.3) we can get

$$\lambda^i{}_{,\alpha\beta\gamma} = (M^{\theta}{}_{\alpha\beta,\gamma} - N_{\alpha\beta}\, d_{\gamma\delta}\, g^{\delta\theta})\, x^i{}_{,\theta} + X^i\, (M^{\theta}{}_{\alpha\beta}\, d_{\theta\gamma} + N_{\alpha\beta,\gamma}) \quad \ldots (2.6)$$

Such that

$$q\{M^{\theta}{}_{\alpha\beta}\, (q\, d_{\theta\gamma} + \mu_{\theta\gamma} - p_{\theta,\gamma}) + M^{\theta}{}_{\beta\gamma}\, \mu_{\theta\alpha} - M_{\beta\theta},\gamma\, \mu^{\theta}{}_{\alpha}\}$$
$$- \mu^{\theta}{}_{\alpha}\, \mu_{\varphi}{}^{\beta}\, d_{\theta\gamma}\, p_{\varphi} - N_{\alpha\beta}\, (\mu^{\theta}{}_{\gamma}\, p_{\theta} + q\, p^{\theta}\, d_{\theta\gamma} + q\, q_{,\gamma}) = 0 \quad \ldots (2.7)$$

The intrinsic derivative of $\frac{d^2\lambda^i}{ds^2}$ along C which is represented by λ'''^i can be obtained in the following form

$$\lambda'''^i = (x^i{}_{,\gamma}\, A^{\gamma} + B\, X^i) \quad \ldots (2.8)$$

where $A^{\gamma} = [\mu^{\gamma}{}_{\alpha}\, u'''^{\alpha} + \mu'''\alpha\, u'\beta\, (2M^{\gamma}{}_{\alpha\beta} + M^{\gamma}{}_{\beta\alpha}) + u'^{\alpha}\, u'^{\beta}\, u'^{\delta}\, (M^{\gamma}{}_{\alpha\beta,\delta} - N_{\alpha\beta}\, d^{\gamma}{}_{\delta})]$ … (2.9)

and $B = [v_{\alpha}\, u'''^{\alpha} + u''^{\alpha}\, u'^{\beta}\, (2N_{\alpha\beta} + N_{\beta\alpha}) + u'^{\alpha}\, u'^{\beta}\, u'^{\delta}\, (d_{\gamma\delta}\, M^{\gamma}{}_{\alpha\beta} + N_{\alpha\beta,\delta})]$ … (2.10)

λ_0–Curves

Definition 3.1: A curve C on the surface S hall be called λ_0–curve in an Euclidean space of three dimensions if the osculating plane at any point P of C contains the vector λ'''^i.

The equation of osculating plane at any point P of C can be written as [1]

$$\delta^{123}_{i\,j\,k}\,('x^i - x^i)\,(u'^{\sigma}\,x^j_{,\sigma})\,(\rho^{\alpha}\,x^k_{,\alpha} + X^k\,k_n) = 0 \qquad \ldots (3.1)$$

where $'x^i$ are current coordinates of a point in Euclidean space of three dimensions and ρ^{α}, the first curvature vector is expressed as [1]

$$\rho^{\alpha} = u''^{\alpha} + \{^{\alpha}{}_{\beta\gamma}\}\,u'^{\beta}\,u'^{\gamma} \qquad \ldots (3.2)$$

Now if λ'''^i lies in the plane (3.1), equation $'x^i = x^i + t\,\lambda'''^i$, must hold for all t. Hence we get

$$\delta^{123}_{i\,j\,k}\,(u'^{\sigma}\,x^j_{,\sigma})\,(\rho^{\alpha}\,x^k_{,\alpha} + X^k\,k_{\alpha})\,(x^i_{,\gamma}\,A^{\gamma} + B\,X^i) = 0 \qquad \ldots (3.3)$$

Using [1]

$$\delta^{123}_{i\,j\,k}\,X^i\,x^j_{,\sigma}\,X^k = 0,\;\; \delta^{123}_{i\,j\,k}\,x^i_{,\gamma}\,x^i_{,\sigma}\,x^k_{,\alpha} = 0 \qquad \ldots (3.4)$$

In equation (3.3), we obtain on simplification

$$\delta^{123}_{i\,j\,k}\,X^i,\,x^j_{,\sigma}\,u'^{\sigma}\,x^k_{,\alpha}\,(B\,\rho^{\alpha} - k_n\,A^{\alpha}) = 0 \qquad \ldots (3.5)$$

Summing (3.5) for σ and α and neglecting non–zero terms and using $e_{12} = -\,e_{21} = 1$ and $e_{11} = e_{22} = 0$, we obtain

$$e_{\sigma\,\alpha}\,u'^{\sigma}\,(B\,\rho^{\alpha} - k_n\,A^{\alpha}) = 0 \qquad \ldots (3.6)$$

Hence we have:

Theorem 3.1: The differential equation of λ_0–curves, in an Euclidean space of three dimensions is given by equation (3.6)

If either C be a geodesic or $B = 0$, the equation (3.6) reduces to

$$e_{\sigma\,\alpha}\,u'^{\sigma}\,A^{\alpha} = 0 \qquad \ldots (3.7)$$

which implies that either $k_n = 0$ or

$$e_{\sigma\,\alpha}\,u'^{\sigma}\,A^{\alpha} = 0 \qquad \ldots (3.8)$$

Using equation (3.2) for a geodesic curve, equation (3.8) can be represented by

$$\sigma_{\alpha\gamma}\,u'^{\sigma}\,u'^{\theta}\,u'^{\varphi}\,u'^{\delta}\,[2\{^{\alpha}{}_{\beta\delta}\}\,\{^{\beta}{}_{\theta\varphi}\}\,\mu^{\gamma}{}_{\alpha} - \mu^{\gamma}{}_{\beta}\,\{^{\beta}{}_{\theta\varphi}\}_{,\delta}$$
$$-\{^{\alpha}{}_{\theta\varphi}\}\,({}^{\mu}\gamma_{\alpha,\delta} + M^{\gamma}{}_{\alpha\delta} + M^{\gamma}{}_{\delta\alpha} - d_{\delta\beta}\,g^{\beta\gamma}\,\nu_{\alpha}) + M^{\gamma}{}_{\theta\varphi,\delta} - N_{\theta\varphi}\,d_{\delta\alpha}\,g^{\alpha\gamma}] = 0 \qquad \ldots (3.9)$$

If we consider that the λ_0–curves are either asymptotic lines or satisfy equation (3.8), equation (3.6) leads to $e_{\sigma\,\alpha}\,u'^{\sigma}\,B\,\rho^{\alpha} = 0$, *i.e.*, either it is a geodesic curve or $B = 0$, *i.e.*, it is a generalized Darboux curve Rastogi and Bajpai [2]. Hence we have:

Theorem 3.2: The necessary and sufficient condition for a λ_0–curve to be either a geodesic curve *i.e.*, $u'^1\,\rho^2 = u'^2\,\rho^1$ or be a generalized Darboux curve, is that it is either an asymptotic line or satisfies equation (3.8).

λ_0–CURVATURE OF A CURVE ON *S*

The equation (3.6) is the single differential equation of λ_0–curves. The equation of λ_0–curves can alternatively be expressed as

$$T^1 \equiv B\,\rho^1 + k_n\,e_{\sigma\,\alpha}\,u'^{\sigma}\,A^{\alpha}\,g_{2\,\beta}\,u'^{\beta} = 0 \qquad \ldots (4.1)\,(i)$$

$$T^2 \equiv B\,\rho^2 - k_n\,e_{\sigma\,\alpha}\,u'^{\sigma}\,A^{\alpha}\,g_{1\,\beta}\,u'^{\beta} = 0 \qquad \ldots (4.1)\,(ii)$$

In analogy with the definition of union curvature [3], we define the vector with contravariant components T^{α} as the λ_0–curvature vector and the magnitude of this vector shall be called λ_0–curvature. From equation (3.1), we can observe the following

Theorem 4.1: The λ_0–curvature vector in an Euclidean space of three dimensions is a null vector at each point of the λ_0–curve.

Using $\varepsilon_{\alpha\beta} = (x^i_{,\alpha}\, x^i_{,\beta}\, X^i) = g\, e_{\alpha\beta}$, the λ_0–curvature of the curve is defined as

$$K_0 = \varepsilon_{\alpha\beta}\, u'^{\alpha}\, T^{\beta} \qquad \dots (4.2)$$

Since $\varepsilon_{\alpha\beta}\, u'^{\alpha}\, u'^{\beta} = 0$, therefore with the help of equation (4.1), we can write (4.2) as

$$K_0 = \varepsilon_{\alpha\beta}\, u'^{\alpha}\, (B\, \rho^{\beta} - k_n\, A^{\beta}) \qquad \dots (4.3)$$

From equation (4.3), we can easily obtain.

Theorem 4.2: The ratio of K_0–curvature of a generalized Darboux curve and normal curvature to the surface is given by $\varepsilon_{\alpha\beta}\, u'^{\alpha}\, A^{\beta}$.

If $k_g = \varepsilon_{\alpha\beta}\, u'^{\alpha}\, \rho^{\beta}$, be the geodesic curvature, from equation (4.3), we can observe that the geodesic curvature along a λ_0–curve ($K_0 = 0$), is given by

$$B\, k_g = k_n\, \varepsilon_{\alpha\beta}\, u'^{\alpha}\, A^{\beta} \qquad \dots (4.4)$$

which is analogy to the geometrical interpretation of union curvature, Springer [3], gives

Theorem 4.3: The K_0–curvature of a curve C at any point P on a surface of reference S of a rectilinear congruence is the curvature of C obtain by projecting C onto the tangent plane to S at P, in the direction of λ'''^{i}.

CURVATURE OF A λ_0–CURVE

Let α^i, β^i and γ^i be respectively the direction cosines of unit tangent, principal normal and binormal to a λ_0–curve C, then we can express β^i as

$$\beta^i = a\, \alpha^i + b\, \lambda'''^{i} \qquad \dots (5.1)$$

Let ψ be the angle between the vectors α^i and λ'''^{i}, then for $D\, |\lambda'''^{i}| = 1$,

$$a = -\cot\psi,\ b = D \operatorname{cosec}\psi \qquad \dots (5.2)$$

From equations (5.1) and (5.2), we get

$$\beta^i = \operatorname{cosec}\psi\, (D\, \lambda'''^{i} - \alpha^i \cos\psi) \qquad \dots (5.3)$$

Since we know that $x''^{i} = k\, \beta^i$, therefore substituting from equation (1.3), we can write

$$\rho^{\alpha}\, x^i_{,\alpha} + X^i\, k_n = k \operatorname{cosec}\psi\, [D\, (x^i_{,\alpha}\, A^{\alpha} + B\, X^i) - x^i_{,\alpha}\, u'^{\alpha} \cos\psi] \qquad \dots (5.4)$$

Multiplying equation (5.4) by X^i, $g^{\delta\beta}\, x^i_{,\delta}\, \varepsilon_{\tau\beta}\, u'^{\tau}$ and λ'''^{i} respectively and solving, we get the following expressions for the curvature k of a λ_0–curve

$$k = (BD)^{-1}\, k_n \sin\psi \qquad \dots (5.5)$$

$$k = (D\, \varepsilon_{\tau\beta}\, u'^{\tau}\, A^{\beta})^{-1}\, k_g \sin\psi \qquad \dots (5.6)$$

and

$$k = \sin\psi\, (\rho^{\alpha}\, A^{\alpha} + B\, k_n)\, [A^{\alpha}\, (DA^{\alpha} - u'^{\alpha} \cos\psi) + DB^2]^{-1} \qquad (5.7)$$

Multiplying equation (5.1) by β_i, we can obtain on simplification

$$k = b\, (\rho^{\alpha}\, A^{\alpha} + B\, k_n) \qquad \dots (5.8)$$

Substituting the value of b from equation (5.2), we get

$$k = D \operatorname{cosec} \psi \, (\rho^{\alpha} A^{\alpha} + B\, k_n) \qquad \ldots (5.9)$$

Since D and cosec ψ can not vanish, therefore from equation (5.9), we can obtain

Theorem 5.1: The necessary and sufficient condition for the curvature k of a λ_0–curve to vanish is given by the vanishing of $p^{\alpha} A_{\alpha} + B\, k_n$

Since we know that $-(k\alpha^i + \tau\gamma^i) = k^{-1} x'''^i$, therefore using equation (1.5), we can obtain

$$`\gamma^I = -(k\tau)^{-1} u'^{\beta} \{A^{\alpha}(\rho^{\alpha}{}_{,\beta} - k_n d^{\alpha}{}_{\beta})\, x^i{}_{,\alpha} + B\,(K_{n,\beta} + \rho^{\alpha} d_{\alpha\beta} X^i\} - k\tau^{-1} x^i{}_{,\alpha} u'^{\alpha} \qquad \ldots (5.10)$$

Which leads to

$$K^2 = -[u'^{\beta} \{A_{\alpha}(\rho^{\alpha}{}_{,\beta} - k_n d^{\alpha}{}_{\beta}) + B\,(k_{n,\beta} + \rho^{\alpha} d_{\alpha\beta})\}/(A_{\alpha} u'^{\alpha}) \qquad \ldots (5.11)$$

Applying Theorem 5.1, to equation (5.11), we obtain on simplification

$$\{\rho^{\alpha}(A_{\alpha,\beta} - B\, d_{\alpha\beta}) + k_n\,(B_{,\beta} + A_{\alpha} d^{\alpha}{}_{\beta})\}\, u'^{\beta} = 0 \qquad \ldots (5.12)$$

Hence we have:

Theorem 5.2: The necessary and sufficient condition for the curvature k of a λ_0–curve to vanish is given by (5.12).

TORSION OF A λ_0–CURVE

Differentiating the identity $\gamma^i = \alpha^i x \beta^I$, with respect to s and using equation (5.3) and the Frenet formula $d\gamma^i/ds = \tau\beta^I$ Eisenhart [1], we get

$$\tau\,(D\lambda'''^I - \cos\psi\, dx^i/ds) = \{(dx^i/ds) x \lambda'''^i\}\,\{D' - D\cot\psi\,(d\psi/ds)\} + D\{d^2x^i/ds^2 x\lambda'''^i + (dx^i/ds) x \lambda^{(4)i}\} \ldots (6.1)$$

Substituting the values of λ''''^I and $d^2 x^i/ds^2$ in equation (6.1) and multiplying the resulting equation X^i, we obtain on simplification the torsion of a λ_0–curve as follows:

$$\tau\,(BD)^{-1}\varepsilon_{\alpha\beta}\,[\{D' - D\cot\psi\,(d\psi/ds)\}\, A^{\beta} u'^{\alpha} + D\{A^{\beta}\rho^{\alpha} + (A^{\alpha}{}_{,\theta} - B\, d^{\alpha}{}_{\theta})\, u'^{\theta} u'^{\beta}\}]. \ldots (6.2)$$

With the help of equation (5.6) and (6.2), we can obtain a relationship between the curvature k and the torsion τ.

SOME SPECIAL CASES

Cash I: Normal Congruence. Let us consider that the congruence be normal, then in that case we have $p_{\alpha} = 0$, $q = 1$, $\lambda^i = X^i$, $\mu^{\gamma}{}_{\alpha}, = -d^{\gamma}{}_{\alpha}\, v_{\alpha} = 0$, $M^{\gamma}{}_{\alpha\beta} = -d^{\gamma}{}_{\alpha,\beta}$, $N_{\alpha\beta} = -d^{\gamma}{}_{\alpha}\, d_{\gamma\beta}$ and equation (2.6) can be expressed as

$$e_{\sigma\alpha} u'^{\sigma}\,[k_n u'''^{\gamma} d^{\alpha}{}_{\gamma} - u''^{\delta} u'^{\beta}\,\{\rho^{\alpha}(2d^{\gamma}{}_{\delta}\, d_{\gamma\beta} + d^{\gamma}{}_{\beta}\, d^{\gamma}{}_{\delta}) - k_n\,(2d^{\alpha}{}_{\delta,\beta} + d^{\alpha}{}_{\beta,\delta})\} - u'^{\theta} u'^{\beta} u'^{\delta}\,\{\rho^{\alpha}(d_{\gamma\delta}\, d^{\gamma}{}_{\theta,\beta} + d^{\gamma}{}_{\theta}\, d_{\gamma\beta,\delta} + d^{\gamma}{}_{\theta,\delta}\, d_{\gamma\beta}) - k_n\,(d^{\alpha}{}_{\theta,\beta,\delta} - d^{\alpha}{}_{\delta}\, d^{\gamma}{}_{\theta}\, d_{\gamma\beta})\}] \qquad \ldots (6.3)$$

Hence we have

Theorem 7.1: For a normal congruence in an Euclidean space of three dimensions; λ_0–curves satisfy equation (7.1).

Case II : Congruence formed by tangents to a one parameter family of curves. In such a case, we have $q = 0$, ρ^α is a unit vector, $\lambda^i = x^i,_\alpha \rho^\alpha$, $u^\gamma{}_\alpha = \rho^\gamma{}_\alpha$, $v_\alpha = \rho^\beta d_{\alpha\beta}$ and hence equation (3.6) can be expressed as

$$e_{\sigma\alpha} u'^\sigma [u'''^\gamma (d_{\gamma\beta} \rho^\alpha p^\beta - k_n \rho^\alpha{}_{,\gamma}) + u''^\delta u'^\beta \{\rho^\alpha (2\rho^\gamma{}_{,\delta} d_{\gamma\beta} + 2(\rho^\theta d_{\delta\theta}),^\beta$$
$$+ \rho^\gamma{}_{,\beta} d_{\gamma\delta} + (\rho^\theta \cdot d_{\beta\theta}),_\delta) - k_n (2\rho^\alpha,_{\delta,\beta} - 2\rho^\theta d_{\delta\theta} d^\alpha{}_\beta + \rho^\alpha{}_{,\beta,\delta}$$
$$\rho^\theta d_{\beta\theta} d^\alpha{}_\delta)\} + u'^\varphi u'^\beta u'^\delta \{\{\rho^\alpha d_{\gamma\delta} (\rho^\gamma,_{\varphi,\beta} - \rho^\theta d_{\varphi\theta} d^\gamma{}_\beta) + \rho^\alpha(\rho^\gamma,_{\varphi,\delta} d_{\gamma\beta}$$
$$+ \rho^\gamma{}_{,\varphi} d_{\gamma\beta,\delta} + (\rho^\theta d_{\varphi\theta})_{,\beta,\delta})\} - k_n\{\rho^\alpha{}_{,\varphi,\beta,\delta} - \rho^\theta{}_{,\delta} d_{\varphi\theta} d^\alpha{}_\beta - \rho^\theta d_{\varphi\theta,\delta}$$
$$d^\alpha{}_\beta - \rho^\theta \cdot d_{\varphi\theta} d^\alpha{}_{\beta,\delta} - d^\alpha{}_\delta (\rho\theta, \varphi\, d\varphi\beta + (\rho^\theta d_{\varphi\theta}),_\beta)\}\}] = 0 \quad \ldots (6.4)$$

Hence we have:

Theorem 7.2: In an Euclidean space of three dimensions, for a congruence formed by tangents to a one parameter family of curves, λ_0–curves satisfy equation (7.2).

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