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SOME PATTERNED CONSTRUCTION OF GROUP-DIVISIBLE AND RECTANGULAR DESIGNS

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Some series of group-divisible and rectangular designs have been constructed through generalized row orthogonal constant column matrices (GROCM).Some group-divisible designs listed in Clatworthy's table have been constructed using these series. Some rectangular designs in the range of r,k≤10 have been constructed which may be new.

KEYWORDS: Hadamard Matrix; Difference Matrix; Association Scheme; Partially balanced incomplete block designs; Rectangular designs; Group-divisible designs; Circulant Matrix; GROCM.

INTRODUCTION

We recall the following definitions from Dey [5]

1.1. Difference Matrix - Let λ (≥ 1) and m, n (≥ 2) be positive integers and G be a finite, additive abelian group consisting of m elements. Then a difference matrix D (λm, n; m) is a $\lambda m \times n$ matrix with elements from G such that among the differences of the corresponding elements of every two distinct columns, each element of G appears λ times.

1.2. Hadamard Matrix - A square matrix H of order n with entries from $\{1, -1\}$ is called a Hadamard matrix if $HH^T = nI_n$ where I_n is an $n \times n$ identity matrix.

1.3. Balanced Incomplete Block Design (BIBD)

Let $V = \{1, 2, 3, \ldots, v\}$ be a non-empty set and $\beta = \{\beta_1, \beta_2, \ldots, \beta_b\}$ be a multi set of subsets of V. Then (V, β) is a block design. The elements of V are called treatments and the elements of β are called blocks.

Let $V = \{1, 2, 3, \ldots, v\}$ be a non-empty set and $\beta = \{\beta_1, \beta_2, \ldots, \beta_b\}$ be a multi set of subsets of V. The elements of V are called treatments and the elements of β are called blocks. A BIBD is an arrangement of v treatments into b blocks such that each block contains k treatments, each treatment belongs to r blocks and each pair of treatments belongs to λ blocks. v, b, r, k, λ are called parameters of the BIBD. These parameters are not all independent but are related by the following relations:

(i) $vr = bk$ (ii) $r(k-1) = \lambda(v-1)$.

A BIBD for which $v = b$ is called a Symmetric BIBD (SBIBD).

1.4. Circulant Matrix

An $n \times n$ matrix $C = [c_{ij}]_{0 \le i,j \le n-1}$ where $c_{ij} = c_{j-i(\text{mod} n)}$ is a circulant matrix of order n.

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}$$

$$= \text{circ}\,(c_0, c_1, \dots, c_{n-1})$$

1.5. *m*-class Association scheme (AS) and Association matrices

Let X be a non-empty set of order v. A set $\Omega = \{R_0, R_1, \dots, R_m\}$ of non-empty relations on X is an m-class AS if following properties are satisfied

(i) $R_0 = \{(x,x)\} : x \in X\}$

(ii) Ω is a partition of $X \times X$ *i.e.*

$$\bigcup_{i=0}^{m} R_i = X \times X, R_i \cap R_j = \phi \text{ if } I \neq j.$$

(iii) $R_i^T = R_i$ where $R_i^T = \{(x,y): (y,x) \in R_i\}, i = 0,1,\dots,m.$

(iv) Let $(x,y) \in R_i$. For $i,j,k \in 0,1,\dots,m$

$$p_{jk}^i = p_{kj}^i = \left|\{z: (x,z) \in R_j \bigcap (z,y) \in R_k\}\right|$$

which is independent of $(x,y) \in R_i$.

If $(x,y) \in R_i$ then x and y are called ith associates. For a given treatment $\alpha \in X$, the number of treatments which are i-th associates of α is n_i $(i = 0, 1, 2, \dots, m)$, where the number n_i is independent of the treatment α chosen. The non-negative integers v, n_i, p_{jk}^i $(i, k, j = 0, 1, \dots, m)$ are called the parameters of the m-Class AS. Every treatment is zero-th associate of itself. These parameters are not all independent but are connected by the following relations

(i) $\sum_{i=1}^{m} n_i = v - 1$

(ii) $\sum_{k=1}^{m} p_{jk}^i = \begin{cases} n_j - 1 \text{ if } i = j \\ n_j \text{ if } i \neq j \end{cases}$

(iii) $n_i p_{jk}^i = n_j p_{ik}^j$

Association matrices were introduced by Bose and Mesner [2].

The i-th association matrix $B_i = [b_{\alpha\beta}^i]_{\substack{0 \le i \le m \\ \alpha,\beta \in X}}$ of an m-class AS is a symmetric matrix of order v where

$$b_{\alpha\beta}^i = \begin{cases} 1 \text{ if } \alpha \text{ and } \beta \text{ are mutually ith associates} \\ 0 \text{ Otherwise} \end{cases}$$

Association Matrices have the following properties

(i) $B_0 = I_v$ (ii) $\sum_{i=0}^{m} B_i = J_v$ (iii) $B_i B_j = B_j B_i = \sum_{k=0}^{m} p_{ij}^k$ $(i,j = 0,1,2,\dots,m)$

1.6. Partially Balanced Incomplete Block (PBIB) Design

Let X be a non-empty set with cardinality v. The elements of X are called treatments. A PBIB design based on an m-class association scheme is a family of b subsets (blocks) of X, each of size k such that each treatment occurs in r blocks, any two treatments occur together in

λ_i (i = 0, 1, …, m) blocks if they are mutually i^{th} associates. v, b, r, k, λ_i are called parameters of a PBIB design.

The following relations connect these parameters of PBIB design and also of the parent association scheme:

(i) $vr = bk$ (ii) $\sum_{i=0}^{m} n_i \lambda_i = rk$, where $\lambda_0 = r$.

1.7. Group divisible (GD) and Rectangular AS

A GD AS is an arrangement of $v = mn$ treatments in a rectangular array of m rows and n columns such that any two treatments belonging to the same row are first associates and remaining pairs of treatments are second associates.

The parameters of GD AS are as follows:

$$v = mn,\ n_1 = n - 1,\ n_2 = n\,(m - 1)$$

$$P_1 = \begin{bmatrix} n-2 & 0 \\ 0 & n(m-1) \end{bmatrix},\ P_2 = \begin{bmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{bmatrix}$$

Rectangular AS, introduced by Vartak [22] , is an arrangement of $v = mn$ treatments in a rectangular array of m rows and n columns such that any two treatments belonging to the same row are first associates, any two treatments belonging to the same column are second associates and remaining pairs of treatments are third associates.

1.8. Group divisible design (GDD)

A GDD is a 2-class PBIB design based on a GD AS of $v = mn$ treatments arranged with b blocks such that each block contains k distinct treatments, each treatment occurs in exactly r blocks and any two treatments which are first associates occur together in λ_1 blocks, whereas any two treatments which are second associates occur together in λ_2 blocks. v, b, r, k, λ_1, λ_2 are called the parameters of the GDD .

Let N be the incidence matrix of a GD design with parameters $v = mn$, b, r, k, λ_1, λ_2. Then the eigen values (θ_i) and the corresponding multiplicities (α_i) of the matrix NN^T are given by

$$\theta_0 = rk,\ \theta_1 = r - \lambda_1,\ \theta_2 = rk - v\lambda_2,\ \alpha_0 = 1,\ \alpha_1 = m\,(n - 1),\ \alpha_2 = m - 1.$$

GD designs have been classified into following three categories based on the eigen values of NN^T

(i) Singular, if $r = \lambda_1$

(ii) Semi-regular, if $r > \lambda_1$ and $rk = v\lambda_2$

(iii) Regular, if $r > \lambda_1$ and $rk > v\lambda_2$.

1.9. Rectangular Design

Rectangular design is a 3-class PBIB design based on a rectangular AS of $v = mn$ treatments arranged in a rectangular array of m rows and n columns in b blocks such that

(i) Each block contains k distinct treatments

(ii) Each treatment occurs in exactly r blocks

(iii) Any two treatments which are first associates occur together in λ_1 blocks, whereas any two treatments which are second associates occur together in λ_2 blocks and the treatment which are third associates occur together in λ_3 blocks. v, b, r, k, λ_1, λ_2, λ_3 are called parameters of the rectangular design.

Rectangular designs have been studied by Suen [21], Sinha [15], Sinha *et al.* [16, 17, 18, 19, 20], Kageyama and Miao [9] and so on. The rectangular designs are useful for factorial experiments, having factorial balance as well as orthogonality [7]. GD designs have been studied by Bose and Connor [2], Dey [4], Dey and Nigam [6], John and Turner [8], Rao [10], Seberry [13]and so on. GD designs are also suitable for factorial experiment [4]. Singh and Prasad [14] defined Generalized Orthogonal Combinatorial matrix (GOCM). Saurabh and Singh [12] defined Generalized Row Orthogonal Matrices (GROM) and Generalized Row Orthogonal Constant Column Matrices (GROCM). It is shown that a GROCM is in general an incidence matrix of certain 2- and 3-associate PBIB designs. In this paper some rectangular designs in the range of $r, k \leq 10$ have been constructed which may be new. Some GD designs listed in Clatworthy's table [3] have been constructed.

For convenience, I_n denotes the identity matrix of order n, $J_{t\times u}$ denotes the $t \times u$ matrix all of whose entries are 1 and $K_{t\times u} = J_{t\times u} - I_{t\times u}$. $A \otimes B$ denotes Kronecker product of matrices A and B. $\alpha^i = \text{circ.}(0,0,0,\ldots 1,\ldots,0)$ is a basiccirculant matrix of order n with 1 at $(I + 1)$-th position such that $\alpha^n = I_n$.

DEFINITION OF GROCM

We recall the definition of GROCM

Let $N = [N_{ij}]$, $i, j \in \{1, 2, \ldots, m\}$ where N_{ij} are $\{0, 1\}$ matrices of order $n \times s_j$. Let $R_i = (N_{i1}, N_{i2}, \ldots, N_{im})$ be the ith row of blocks. We define inner product of two rows of blocks R_i and R_j as $R_i \circ R_j = R_i R_j^T = \sum_{k=1}^{m} N_{ik} N_{jk}^T$.

N is called a Generalized Row Orthogonal Matrices (GROM) if there exists fixed positive integer r and fixed non-negative integers $\lambda_1, \lambda_2, \lambda_3$ such that

$$R_i \circ R_j = R_i R_j^T = \sum_{k=1}^{m} N_{ik} N_{jk}^T = \begin{cases} rI_n + \lambda_1 K_n \text{ if } i = j \\ \lambda_2 I_n + \lambda_3 K_n \text{ if } i \neq j. \end{cases}$$

A $\{0, 1\}$-matrix N is called a constant column matrix if sum of entries in each column of N is constant. A GROM with constant column sum k will be called GROCM.

Clearly $v = mn, b = m(s_1 + s_2 + \cdots + s_m). m, n, s_1, s_2, \ldots, s_m, r, k, \lambda_1, \lambda_2, \lambda_3$ are called the parameters of the GROCM. $\lambda_1, \lambda_2, \lambda_3$ are called concurrences of the GROCM.

CONSTRUCTION THEOREMS

Theorem 3.1. There exists a rectangular design with parameters

$$v = 2p^r, b = 2(p^r + m_1 + m_2), r = p^r + m_1 + m_2, k = p^r, \lambda_1 = 0,$$
$$\lambda_2 = \frac{p^r - 1}{2} + m_1 + m_2,$$
$$\lambda_3 = \frac{p^r + 1}{2}, m = p^r, n = 2.$$

Proof : We know that a Hadamard matrix H of order $p^r + 1$ can be constructed by Paley's type I method, where $p^r + 1 \equiv 0 \pmod 4$. Let H_1 be the core of H. Replace in H_1, 1 and -1 by I_2 and K_2 respectively to obtain a block matrix N_1. Let N_2 denotes m_1 copies of $p^r \times 1$ block matrix with blocks I_2 *i.e.*

$N_2 = \begin{bmatrix} I_2 \\ I_2 \\ \vdots \\ I_2 \end{bmatrix}$ and Let N_3 denotes m_2 copies of $p^r \times 1$ block matrix with blocks K_2 *i.e.* $N_3 = \begin{bmatrix} K_2 \\ K_2 \\ \vdots \\ K_2 \end{bmatrix}$. Then $N = [N_1, N_2, N_3]$ is the incidence matrix of a rectangular design with the required parameters.

Inner product of two rows of N is

$$R_i \circ R_j = \begin{cases} (p^r + m_1 + m_2)I_2 + 0K_2 \text{ if } i = j \\ \left(\frac{p^r - 1}{2} + m_1 + m_2\right) I_2 + \left(\frac{p^r + 1}{2}\right) K_2 \text{ if } i \neq j \end{cases}$$

Corollary 3.1.1 There exists a rectangular design with parameters

$$v = 2(p^r - l), b = 2(p^r + m_1 + m_2), r = p^r + m_1 + m_2, k = p^r - l, \lambda_1 = 0,$$
$$\lambda_2 = \frac{p^r - 1}{2} + m_1 + m_2, \lambda_3 = \frac{p^r + 1}{2}, m = p^r - l, n = 2.$$

Proof: If we delete l rows of blocks of N, then we obtain a rectangular design with the required parameters.

Some rectangular designs in the range of $2 \leq r, k \leq 10$ using the above series which may be new

(1) $v = 14, b = 20, r = 10, k = 7, \lambda_1 = 0, \lambda_2 = 6, \lambda_3 = 4, m = 7, n = 2.$

(2) $v = 12, b = 20, r = 10, k = 6, \lambda_1 = 0, \lambda_2 = 6, \lambda_3 = 4, m = 6, n = 2.$

(3) $v = 10, b = 20, r = 10, k = 5, \lambda_1 = 0, \lambda_2 = 6, \lambda_3 = 4, m = 5, n = 2.$

(4) $v = 8, b = 20, r = 10, k = 4, \lambda_1 = 0, \lambda_2 = 6, \lambda_3 = 4, m = 4, n = 2.$

(5) $v = 14, b = 18, r = 9, k = 7, \lambda_1 = 0, \lambda_2 = 5, \lambda_3 = 4, m = 7, n = 2.$

(6) $v = 12, b = 18, r = 9, k = 6, \lambda_1 = 0, \lambda_2 = 5, \lambda_3 = 4, m = 6, n = 2.$

(7) $v = 10, b = 18, r = 9, k = 5, \lambda_1 = 0, \lambda_2 = 5, \lambda_3 = 4, m = 5, n = 2.$

(8) $v = 8, b = 18, r = 9, k = 4, \lambda_1 = 0, \lambda_2 = 5, \lambda_3 = 4, m = 4, n = 2.$

Theorem 3.2. There exists a Semiregular GD design with parameters

$$v = 2p^r, b = 2(p^r + 1), r = p^r + 1, k = p^r, \lambda_1 = 0, \lambda_2 = \frac{p^r + 1}{2}, m = p^r, n = 2.$$

Proof: Saurah and Singh [11] have shown that a GROCM N represents a GD design if either $\lambda_1 = \lambda_3$ or $\lambda_2 = \lambda_3$. Hence N represents a GD design if $m_1 + m_2 = 1$ and thus we obtain a GD design with the required parameters.

Corollary 3.2.1. There exists a Semiregular GD design with parameters

$$v = 2(p^r - s), b = 2(p^r + 1), r = p^r + 1, k = p^r - s, \lambda_1 = 0,$$
$$\lambda_2 = \frac{p^r + 1}{2}, m = p^r - s, n = 2.$$

Proof : If we delete s rows of blocks from the incidence matrix of the previous GD design then we obtain a GD design with the required parameters.

Particular cases are SR2, SR4, SR19, SR21, SR39, SR54, SR69, SR82.

Theorem 3.3. Let λ (≥ 2) and t (≥ 2) be powers of the same prime then a Semiregular GD design with parameters $v = b = \lambda m^2, r = k = \lambda m, \lambda_1 = 0, \lambda_2 = \lambda, m = \lambda t, n = m$ can be constructed using $D\ (\lambda m, \lambda m; m)$.

Proof: The proof is an easy verification.

Particular cases are SR67, SR72, SR92, SR95, SR102, SR108.

CONCLUSION

A combinatorial structure GROCM has been used to construct some certain 2- and 3-associate PBIB designs. A GROCM can also be used to construct some more combinatorial designs which are suitable for factorial experiments.

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REFERENCES

1. Bose, R.C., Connor, W.S., Combinatorial properties of group divisible incomplete block designs, *Ann. Math. Statist.*, **23**, 367-383 (1952).
2. Bose, R.C., Mesner, D.M., On linear associative algebras corresponding to association schemes of partially balanced designs, *Ann. Math. Statist.*, **23**, 21-38 (1959).
3. Clatworthy, W.H., Tables of two-associate Partially Balanced Designs, *National Bureau of Standards. Applied Maths.*, Series No. **63**, Washington D.C. (1973).
4. Dey, A., Construction of regular group divisible designs, Biometrika, **64**, 647-649 (1977).
5. Dey, A., Theory of block designs, New York, Wiley Eastern (1986).
6. Dey, A., Nigam, A.K., Construction of group divisible designs, *J. Indian Soc. Agric. Statist.*, **37(2)**, 163-166 (1985).
7. John, J.A., Dean, A.M., Single Replicate Factorial Experiments in Generalized Cyclic Designs : I. Symmetrical Arrangements, *Wiley for the Royal Society*, **37**, 63-71 (1975).
8. John, J.A., Turner G., Some new group divisible designs, *J. Statist. Plan. Inference*, **1**, 103-107 (1977).
9. Kagyeyama, S., Miao, Y., Some methods of constructions of rectangular PBIB Designs, *Combinatorics Theory*, **1**, 186-198 (1995).
10. Rao, M.B., Group divisible family of PBIB designs, *J. Indian Statist. Assoc.*, **4**, 14-28 (1966).
11. Saurabh, S., Singh, M.K., Construction of some combinatorial designs from Generalized Row Orthogonal Constant Column Matrices, *Acta Ciencia Indica*, **3**, 181-187 (2017).
12. Saurabh, S., Singh, M.K., Construction of Rectangular designs from Generalized Orthogonal Constant Column Matrices, *International Journal of Mathematical Archive*, **8(11)**, 135-139 (2017).
13. Seberry, J., Regular group divisible design and Bhaskar Rao designs with block size 3, *J. Statist. Plann. and Inference*, **10**, 69-82 (1984).
14. Singh, M.K., Prasad, D., Some new balanced block intersection designs from Williamson matrices, *Acta Ciencia Indica*, **3**, 371-376 (2014).
15. Sinha, K., A construction of rectangular designs. *J. Comb. Math. Comb. Computing*, **9**, 199-200 (1991).
16. Kagyeyama, S., Sinha, K., Some patterned construction of rectangular designs, *J. Japan Statist. Soc.*, **33**, 137-144 (2003).
17. Sinha, K., Kagyeyama, S., Singh, M.K., Construction of rectangular designs, *Statistics*, **25**, 63-70 (1993).

18. Sinha, K., Mitra, R.K. and Saham G.M., Nested BIB designs, balanced bipartite weighing designs and rectangular designs, *Utilitas Math.*, **49**, 216-22 (1996).
19. Sinha, K., Mitra, R.K., Constructions of nested balanced incomplete block designs. rectangular designs and q-ary codes, *Ann. Combin.*, **3**, 71-80 (1999).
20. Sinha, K., Singh, M.K., Kagyeyamam S., Singh, R.S., Some series of rectangular Designs, *J. Statist. Plann. Inference*, **106**, 39-46 (2002b).
21. Suen, C., Some rectangular designs constructed by method of differences, *J. Statist. Plann. Inference,* **21**, 273-276 (1989).
22. Vartak, M.N., On an application of Kronecker product of matrices to statistical designs, *Ann. Math. Statist.*, **26**, 420-438 (1955).

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