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ON POLYNOMIAL SOLUTIONS OF QUADRATIC DIOPHANTINE EQUATION

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Let $P = P(t)$ be a polynomial in $Z[x]$. In this paper, we consider the polynomial solutions of Diophantine equation $D : w^2 - 20z^2 - 4w - 120z - 180 = 0$. We also obtain some formulae and recurrence relations on the polynomial solution (w_n, z_n) of D.

KEYWORDS: Diophantine equation, polynomial solution, Pell's equation, Continued fraction expansion.

INTRODUCTION

A Diophantine equation is a polynomial equation $P(x_1, x_2, x_3, \text{-----}\ x_n) = 0$ where the polynomial P has the integral coefficients and one is interested in solutions for which all unknowns take integer values. For example, $x^2 + y^2 = z^2$ and $x = 6$, $y = 8$, $z = 10$ is one of its infinitely many solutions. Another example is $x + y = 1$ and all its solutions are given by $x = s$, $y = 1 - s$ where s passes through all integers. A third example $x^2 + 4y = 5$ the Diophantine equation has no solutions, although note that $x = 0$, $y = \frac{5}{4}$ is a solution with rational values for the unknowns. Diophantine equations are rich in variety. Two-variable Diophantine equation have been a subject to extensive research, and their theory constitutes one of the most beautiful, most elaborate part of mathematics, which nevertheless still keeps some of its secrets for the next generation of researchers.

In this paper we investigate positive integral solutions of the Diophantine equation $w^2 - 20z^2 - 4w - 120z - 180 = 0$ which is transformed into Pell's equation and it solved by various methods.

THE DIOPHANTINE EQUATION $w^2 - 20z^2 - 4w - 120z - 180 = 0$

In consider the Diophantine equation

$$D : w^2 - 20z^2 - 4w - 120z - 180 = 0 \qquad \ldots (1)$$

To be solved over Z. It is not easy to solve and find the nature and properties of the solutions of (1). So we apply a linear transformation D to (1) to transform to a simpler from for which we can determine the integral solutions.

Let $$T : \begin{cases} w = x + h \\ z = y + k \end{cases} \quad \ldots (2)$$

be the transformation where $h, k \in Z$.

Applying T to D, we get

$$T(D) : \widetilde{D}\ (x+h)^2 - 20\,(y+k)^2 - 4\,(x+h) - 120\,(y+k) = 180 \quad \ldots (3)$$

Equating the coefficients of x and y to zero, we get $h = 2$, $k = -3$. Hence for $w = x + 2$ and $z = y - 3$, we have the Diophantine equation

$$\widetilde{D}: x^2 - 20y^2 = 4 \quad \ldots (4)$$

Which is the Pell equation, Now we try to find all integer solutions (x_n, y_n) of $\widetilde{D}$ and then we can retransfer all results from $\widetilde{D}$ to D by using the inverse of T.

Theorem 2.1 : Let $\widetilde{D}$ be the Diophantine equation in (4) then

(i) The continued fraction expansion of $\sqrt{20}$ is $\sqrt{20} = [4 : \overline{2, 8}]$

(ii) The fundamental solution of $x^2 - 20y^2 = 1$ is $(u_1, v_1) = (9, 2)$

(iii) For $n \geq 4$ $\quad u_n = 19\,(u_{n-1} - u_{n-2}) + u_{n-3}$

$v_n = 19\,(v_{n-1} - v_{n-2}) + v_{n-3}$

Proof: (i) The coefficient fraction expansion of $\sqrt{20} = 4 + (\sqrt{20} - 4)$

$$= 4 + \cfrac{1}{\cfrac{1}{\sqrt{20} - 4}}$$

$$= 4 + \cfrac{1}{\cfrac{\sqrt{20} + 4}{4}}$$

$$= 4 + \cfrac{1}{2 + \cfrac{\sqrt{20} - 4}{4}}$$

$$= 4 + \cfrac{1}{2 + \cfrac{\frac{1}{4}}{\sqrt{20} - 4}}$$

$$= 4 + \cfrac{1}{2 + \cfrac{1}{\sqrt{20} + 4}}$$

$$= 4 + \cfrac{1}{2 + \cfrac{1}{8 + (\sqrt{20} - 4)}}$$

Therefore the continued fraction expansion of $\sqrt{20}\ is\ [4,2,8]$

(ii) Note that by (3), if $(u_1, v_1) = (9, 2)$ is the fundamental solution of $x^2 - 20y^2 = 1$ can be derived by using the equalities

$(u_n + v_n\sqrt{20}) = (u_1 + v1_n\,\sqrt{20})^n$ for $n \geq 2$, in other words

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 & 20v_1 \\ 2 & u_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For $n \geq 2$. Therefore it can be shown by induction on n that

$$u_n = 19\,(u_{n-1} - u_{n-2}) + u_{n-3}$$

and also
$$v_n = 19\,(v_{n-1} - v_{n-2}) + v_{n-3}$$

for $n \geq 4$.

Now we consider the problem $x^2 - 20y^2 = 4$

Note that we denote the integer solutions of $x^2 - 20y^2 = 4$ by (x_n, y_n) and denote the integer solutions of $x^2 - 20y^2 = 1$ by (u_n, v_n). Then we have the following theorem.

Theorem 2.2: Define a sequence $\{(x_n, y_n)\}$ of positive integers by

$$(x_1, y_1) = (18, 4)$$

and
$$x_n = 18u_{n-1} + 80v_{n-1}$$
$$y_n = 4u_{n-1} + 18v_{n-1}$$

where $\{(u_n, v_n)\}$ is a sequence of positive solutions of $x^2 - 20y^2 = 1$.

Then (1) (x_n, y_n) is a solution of $x^2 - 20y^2 = 4$ for any integer $n \geq 1$.

(2) For $n \geq 2$ $\quad x_{n+1} = 9x_n + 40y_n$

$y_{n+1} = 2x_n + 9x_n$

(3) For $n \geq 4$ $\quad x_n = 19\,(x_{n-1} - x_{n-2}) + x_{n-3}$

$y_n = 19\,(y_{n-1} - y_{n-2}) + y_{n-3}$

Proof: (1) It is easily seen that

$$(x_1, y_1) = (18, 4)$$

is a solution of $x^2 - 20y^2 = 16$ since

$$x_1^2 - 20y_1^2 = (18)^2 - 20(4)^2$$
$$= 324\text{-}320$$
$$= 4$$

Note that by definition (u_{n-1}, v_{n-1}) is a solution of $x^2 - 20y^2 = 1$, that is

$$u_{n-1}^2 - 20v_{n-1}^2 = 1 \qquad \ldots (5)$$

Also we see as above that (x_1, y_1) is a solution of $x^2 - 20y^2 = 4$, that is

$$x_1^2 - 20y_1^2 = 4 \qquad \ldots (6)$$

Applying (5) and (6),

we get
$$x_n^2 - 20y_n^2 = (18u_{n-1} + 80v_{n-1})^2 - 20\,(4u_{n-1} + 18v_{n-1})^2$$
$$= 2^2\, u_{n-1}^2 - 20\,(2^2)\, v_{n-1}^2$$
$$= 2^2\,(u_{n-1}^2 - 20v_{n-1}^2)$$
$$= 2^2.$$

Therefore (x_n, y_n) is a solution of $x^2 - 20y^2 = 4$

(2) recall that $\quad x_{n+1} + y_{n+1}\sqrt{d} = (u_1 + v_1\sqrt{d})\,(x_n + y_n\sqrt{d})$

Therefore $\quad x_{n+1} = u_1\, x_n + v_1\, y_n\, d$

$y_{n+1} = v_1\, x_n + u_1\, y_n$

So $\quad x_{n+1} = 9x_n + 40y_n$ and $y_{n+1} = 2x_n + 9y_n \qquad \ldots (*)$

since $u_1 = 9$, and $v_1 = 2$

(3) Applying the equalities $x_n = 9u_{n-1} + 40v_{n-1}$ and $x_{n+1} = 9x_n + 40y_n$ we find by induction on n that $x_n = 19\,(x_{n-1} - x_{n-2}) + x_{n-3}$ and for $n \geq 4$, similarly it can be shown that

$$y_n = 19\,(y_{n-1} - y_{n-1}) + y_{n-3}$$

Corollary 2.3 : The base of the above transformation T in (2) is the fundamental solution of D_1, that is $T\,[h, k] = \{x_1, y_1\}$.

Proof : We proved that $(x_1, y_1) = (18, 4)$ is the fundamental solution of D_1. Also we showed that $h = 2$, and $k = -3$. So the base of $T\,[h, k] = \{2, -3\}$ as we claimed .we saw as above that the Diophantine equation D could be transformed into the Diophantine equation D_1 via the transformation T. Also we showed that $w = x + 2$ and $z = y - 3$. So we can retransfer all results from D_1 to D by using the inverse of T. Thus we can give the following main theorem.

Theorem 2.4 : Let the Diophantine equation in (1), then

(i) The fundamental solution of D is $(w_1, z_1) = (20, 1)$

(ii) Define the sequence $\{(w_1, z_1)\}_{n \geq 1} = \{(x_1 + 2, y_1 - 3)\}$, where $\{(x_n, y_n)\}$ defined in (*)

(iii) The solution (w_n, z_n) satisfy

$$w_n = 18w_{n-1} + 80z_{n-1} + 206 \quad \ldots (7)$$

$$z_n = 4w_{n-1} + 18z_{n-1} + 49 \quad \ldots (8)$$

(iv) The solutions (w_n, z_n) satisfy the recurrence relations

$$w_n = 19\,(w_{n-1} - w_{n-2}) + w_{n-3}$$

$$z_n = 19\,(z_{n-1} - z_{n-2}) + z_{n-3}$$

Proof : (i) It is easily seen that $(w_1, z_1) = (20, 1)$ is the fundamental solution of D since $20^2 - 20(1)^2 - 4(20) - 120(1) - 180 = 0$.

(ii) We prove it by induction. Let $n = 1$. Then $(w_1, z_1) = (x_1 + 2, y_1 - 3) = (20, 1)$ which is the fundamental solution and so is a solution of D. Let us assume that the Diophantine equation in (1) is satisfied for $n - 1$, that is

$$(x_{n-1} + 2)^2 - 20(y_{n-1} - 3)^2 - 4\,(x_{n-1} + 2) - 120\,(y_{n-1} - 3) - 180 = 0.$$

We want to show that this equation is also satisfied for n.

$$w^2 - 20z^2 - 4w - 120z - 180 = (x_n + 2)^2 - 20\,(y_n - 3)^2 - 4\,(x_n + 2) - 120\,(y_n - 3) - 180$$

$$= x_n^2 - 20y_n^2 - 4$$

$$= 0 \qquad \text{since } x_n \text{ and } y_n \text{ are solutions of } D_1$$

So $(w_1, z_1) = (x_1 + 2, y_1 - 3)$ is also solution of D.

(iv) From (*) $\quad x_n = 9x_{n-1} + 40y_{n-1}$

Adding 2 on both sides,

$$x_n + 2 = 9x_{n-1} + 40y_{n-1} + 2$$

We know that $\quad w_{n-1} = x_{n-1} + 2$

and $\quad z_{n-1} = y_{n-1} - 3$

therefore $\quad x_{n-1} = w_{n-1} - 2$

and $\quad y_{n-1} = z_{n-1} + 3$

$$x_n = 9x_{n-1} + 40y_{n-1}$$

$$w_n - 2 = 9\,(w_{n-1} - 2) + 40\,(z_{n-1} + 3)$$

$$w_n = 9w_{n-1} + 40z_{n-1} + 104$$

Similarly $$z_n = 4w_{n-1} + 9z_{n-1} + 14$$

We prove that x_n satisfies the recurrence relation. For $n = 4$ we get $w_1 = 20$, $w_2 = 324$, $w_3 = 6476$, and $w_4 = 142164$. Hence

$$w_4 = 19\,(w_3 - w_2) + w_1$$

$$= 19\,(6476 - 324) + 20$$

So $w_4 = 19\,(w_3 - w_2) + w_1$ is satisfied for $n = 4$. Let us assume that this relation is satisfied for $n - 1$, that is

$$w_{n-1} = 19\,(w_{n-2} - w_{n-3}) + w_{n-4} \qquad \ldots (9)$$

Then applying the previous assertion, (7) and (8) we conclude that

$$w_n = 19\,(w_{n-1} - w_{n-2}) + w_{n-3} \text{ for } n \geq 4.$$

Now we prove that z_n satisfied the recurrence relation. For $n = 4$, we get $z_1 = 1$, $z_2 = 89$, $z_3 = 2097$, $z_4 = 44861$. Hence

$$Z_4 = 19\,(z_3 - z_2) + z_1$$

$$= 19\,(2097 - 89) + 1$$

$z_n = 19\,(z_{n-1} - z_{n-2}) + z_{n-3}$ is satisfied for $n = 4$. Let us assume that this relation is satisfied for $n - 1$, that is

$$z_{n-1} = 19\,(z_{n-2} - z_{n-3}) + z_{n-4} \qquad \ldots (10)$$

Then applying the previous assertion, (9) and (10), we conclude that

$$z_n = 19\,(z_{n-1} - z_{n-2}) + z_{n-3}, \text{ for } n \geq 4$$

CONCLUSION

Diophantine equations are rich in variety. There is no universal method for finding all possible solution (if it exists) for Diophantine equations. The methods look to be simple but it is very difficult for reaching the solutions.

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