ON TWO DIOPHANTINE EQUATIONS $2^x + 3^y = z^2$ AND $2^x + 7^y = z^2$

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In this paper, we show that the Two Diophantine Equations $2^x + 3^y = z^2$ and $2^x + 7^y = z^2$ have a unique solution (3, 0, 3) in non-negative integer.

KEYWORDS : Exponential Diophantine Equation, Catalan's Conjecture.

INTRODUCTION

In 2002, J. Sundor studied two Diophantine equations $3^x + 3^y = 6^z$ and $4^x + 18^y = 22^z$. After that D. Acu (2007) studied that the Diophantine equation $2x + 5y = z^2$. He found that this equation has exactly two solutions in non-negative integer $(x, y, z) \in \{(3, 03), (2, 1, 3)\}$. In (2011) A.S. Uvarnamani, A. Singta and S. Chotchaishit studied that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. Inspired by [1] and [3] we study the Diophantine equation $2^x + 3^y = z^2$ and $2^x + 7^y = z^2$ have a common solution (3, 0, 3) where x, y, and z are non negative integers.

MAIN RESULTS

In this study, we use Catalan's Conjecture (see [4]). It is prove there that the only solution in integers $a > 1$, $b > 1$, $x > 1$ and $y > 1$ of the equation $a^x + b^y = 1$ is $a = y = 3$ and $b = x = 2$. Now we have the following theorems

Theorem 2.1: The Diophantine equation $2^x + 3^y = z^2$ has three solutions in non-negative integers.

Proof: From the Diophantine equation

$$2^x + 3^y = z^2 \quad \dots (1)$$

we consider the following four cases

Case (i) : $x = 0$ we have

$$1 + 3^y = z^2$$

Implies that $\quad 3^y = z^2 - 1$

$$3^y = (z - 1)(z + 1)$$

Let $z - 1 = 3^u$ ... (2)

Implies that $3^y = 3^u (z + 1)$

$z + 1 = 3^{y-u}$ ... (3)

(2)-(1) implies $2 = 3^u (3^{y-2u} - 1)$ ... (4)

(3) is true for $u = 0$, $y = 1$.

If $u = 0$ in (1), we get $z - 1 = 1$

$z = 2$

Therefore in this case solution is (0, 1, 2)

Case (ii) : If $y = 0$ we have

$2^x + 1 = z^2$

Implies that $2^x = z^2 - 1$

$2^x = (z - 1)(z + 1)$

Let $z - 1 = 2^u$ ... (5)

We get $z + 1 = 2^{y-u}$ ... (6)

(5)-(4) implies $2 = 2^u [2^{y-2u} - 1]$ ... (7)

If $u = 0$ then there is no y satisfy (7)

It is impossible to find x if $y = 0$.

Case (iii) : If $x \geq 4$, let $x = 4$ in (1) we get

$16 + 3^y = z^2$

Implies $3^y = z^2 - 16$

$3^y = (z - 4)(z + 4)$ ... (8)

Let $z - 4 = 3^v$ ... (9)

We get from (8) $Z + 4 = 3^{y-v}$ ... (10)

(10)-(9) $8 = 3^v [3^{y-2v} - 1]$

If $v = 0$ we get $y = 2$ if $v = 0$ in (9) we get

$Z - 4 = 1, z = 5$

In this case solution for the given equation is (4, 2, 5).

Case (iv) : If $x \geq 3$, let $x = 3$ we get from the equation (1)

$8 + 3^y = z^2$ ... (11)

Implies $3^y = z^2 - 8$

$3^y = (z - 2\sqrt{2})(z + 2\sqrt{2})$

It is not a non negative integer case. So we can find the solution of (11), By trail we can get $y = 0$, $z = 3$.

The solution in this case is (3, 0, 3).

Theorem 2.2: The Diophantine equation $2^x + 7^y = z^2$ has three solutions in non-negative integers.

Proof: From the Diophantine equation $2^x + 7^y = z^2$ ... (12)

we consider the following four cases

Case (i) : $x = 0$ we have

$1 + 7^y = z^2$

Implies that $7^y = z^2 - 1$

$7^y = (z - 1)(z + 1)$

Let $z - 1 = 7^u$ … (13)

Implies that $7^y = 7^u(z + 1)$

$z + 1 = 7^{y-u}$ … (14)

(13)-(14) implies $2 = 7^u(7^{y \quad 2u} - 1)$

It is impossible to find y

Therefore $x \neq 0$

Therefore in this case solution is (0, 1, 2)

Case (ii) : If $y = 0$ we have $2^x + 1 = z^2$

Implies that $2^x = z^2 - 1$

$2^x = (z - 1)(z + 1)$

Let $z - 1 = 2^u$ … (15)

We get $z + 1 = 2^{y-u}$ … (16)

(15)-(16) implies $2 = 2^u[2^{y \quad 2u} - 1]$ … (17)

If $u = 0$ then there is no y satisfy (7)

It is impossible to find x if $y = 0$.

Case (iii) : If $x \geq 5$, let $x = 5$ in (1) we get

$32 + 7^y = z^2$

Implies $7^y = z^2 - 32$

$7^y = (z - 4\sqrt{2})(z + 4\sqrt{2})$ … (18)

It is not a non negative integer case. So we can find the solution of (18), By trail we can get $y = 2$, $z = 9$.

In this case solution for the given equation is (5, 2, 9).

Case (iv) : If $x \geq 3$, let $x = 3$ we get from the equation (1)

$8 + 3^y = z^2$ … (11)

Implies $3^y = z^2 - 8$

$3^y = (z - 2\sqrt{2})(z + 2\sqrt{2})$

It is not a non negative integer case. So we can find the solution of (11). By trail we can get $y = 0$, $z = 3$.

The solution in this case is (3, 0, 3).

CONCLUSION

In this paper we find a common solution for the two Diophantine equations $2^x + 3^y = z^2$ and $2^x + 7^y = z^2$ is (3, 0, 3).

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