HOMOMORPHISM, PARTIAL HOMOMORPHISM AND PAIR HOMOMORPHISM IN BE-ALGEBNRAS

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Here we have discussed some homomorphisms in BE–algebras. Concept of partial homomorphism and pair homomorphism in BE–algebras have been developed with suitable examples and properties.

KEYWORDS : BE–algebra, homomorphism, partial homomorphism, pair homomorphism.

INTRODUCTION

H. S. Kim and Y. N. Kim [1] introduced the concept of BE–algebra in 2006. Since then several related concepts have been developed and studied.

Definition (1.1): A BE–algebra is a system $(X; *, 1)$ consisting of a non-empty set X, a binary operation " $*$ " and a fixed element 1 satisfying the following conditions :-

1. (BE 1) $x * x = 1$
2. (BE 2) $x * 1 = 1$
3. (BE 3) $1 * x = x$
4. (BE 4) $x * (y * z) = y * (x * z)$

for all $x, y, z \in X$.

Definition (1.2): Let $(X; *, 1)$ and $(Y; o, e)$ be BE–algebras and let $f : X \to Y$ be a mapping. Then f is called a homomorphism if $f(x * y) = f(x) \, o \, f(y)$ for all $x, y \in X$.

SOME HOMOMORPHISMS

The following result have been established in [3, 2017].

Theorem (2.1): Let S be an universe and let X be the set of all fuzzy sets defined on S. Let 1^* the 0^* be the fuzzy sets defined on S as

$$1^*(x) = 1 \quad \text{and} \quad 0^*(x) = 0 \text{ for all } x \in S.$$

Also for $f, g \in X$, we define $f = g$ iff $f(x) = g(x)$ for all $x \in S$.

For $f, g \in X$ a binary operation "o" is defined as

$$(f \, o \, g)(x) = \min \{f(x), g(x)\} + 1 - f(x)$$
$$= (f \wedge g)(x) + 1 - f(x).$$

Then $(X; o, 1^*)$ is a BE–algebra with zero element 0^*.

Note (2.2) : For $\alpha \in [0, 1]$, the constant fuzzy set $\alpha(t) = \alpha$ for all $t \in S$ is identified as α.

Definition (2.3) : Let x be a fixed element of S. For $\delta \in X$. Let $\delta(x) = \alpha$. We consider the constant mapping $\alpha : S \to X$ defined as $\alpha(t) = \alpha$ for all $t \in S$. Let $f_x : X \to X$ be defined as

$$f_x(\delta) = \alpha = \delta(x). \quad \ldots (2.1)$$

We prove that

Theorem (2.4): f_x is a homomorphism on X for every $x \in S$.

Proof : Let $\delta_1, \delta_2 \in X$ and let $\delta_1(x) = \alpha$, $\delta_2(x) = \beta$. Then

$$f_x(\delta_1) = \alpha, f_x(\delta_2) = \beta.$$

Let $f_x(\delta_1 \, o \, \delta_2) = \gamma$. Then

$$\gamma(x) = (\delta_1 \, o \, \delta_2)(x)$$
$$= \min \{\delta_1(x), \delta_2(x)\} + 1 - \delta_1(x)$$
$$= \min \{\alpha, \beta\} + 1 - \alpha$$
$$= 1 \text{ or } \beta + 1 - \alpha$$

according as $\alpha \leq \beta$ or $\beta < \alpha$.

This means that

$$f_x(\delta_1 \, o \, \delta_2) = 1^* \text{ or } \beta + 1 - \alpha. \quad \ldots (2.2)$$

according as $\alpha \leq \beta$ or $\beta < \alpha$.

Again, $f_x(\delta_1) \, o \, f_x(\delta_2) = \alpha \, o \, \beta = 1^*$ or $\beta + 1 - \alpha$. … (2.3)

according as $\alpha \leq \beta$ or $\beta < \alpha$.

From (2.2) and (2.3) it follows that

$$f_x(\delta_1 \, o \, \delta_2) = f_x(\delta_1) \, o \, f_x(\delta_2),$$

Hence f_x is a homomorphism .

Example (2.5) : Let $S = \{a, b, c, d, e\}$ and $X = \{\phi, A, B, C, D, E, F, S\}$ where $A = \{a, b\}$, $B = \{a, b, c\}$, $C = \{c\}$, $D = \{c, d, e\}$, $E = \{d, e\}$, $F = \{a, b, d, e\}$; $S \equiv 1$, $\phi \equiv 0$.

For $L, M \in X$, we define a binary operation "o" as

$$L \, o \, M = L^c \cup M.$$

Then Cayley table for this operation 'o' be given by

Table 2.1

o	S	A	B	C	D	E	F	0
S	S	A	B	C	D	E	F	0
A	S	S	S	D	D	D	S	D
B	S	F	S	D	D	E	F	E
C	S	F	S	S	S	F	F	F
D	S	A	B	B	S	F	F	A
E	S	B	B	B	S	S	S	B
F	S	B	B	C	D	D	S	C
0	S	S	S	S	S	S	S	S

Then $(X;\ o, S)$ is a BE–algebra with zero element 0 [5].

For $a \in S$, let $g_a : X \to X$ be a mapping defined as $g_a\,(L)$ = smallest element of X containing L and a.

Then, $g_a\,(A) = A,\ \ g_a\,(B) = B,\ \ g_a\,(C) = B,\ \ g_a\,(D) = S,$

$g_a\,(E) = F,\ \ g_a\,(F) = F,\ \ g_a\,(S) = S,\ \ g_a\,(0) = A.$

Now we see that

$$g_a\,(A\ o\ B) = g_a\,(S) = S,\ \ g_a\,(A)\ o\ g_a\,(B) = A\ o\ B = S;$$
$$g_a\,(A\ o\ D) = g_a\,(D) = S,\ g_a\,(A)\ o\ g_a\,(D) = A\ o\ S = S;$$
$$g_a\,(C\ o\ E) = g_a\,(F) = F,\ \ g_a\,(C)\ o\ \ g_a\,(E) = B\ o\ F = F;$$
$$g_a\,(D\ o\ 0) = g_a\,(A) = A,\ \ g_a\,(D)\ o\ \ g_a\,(0) = S\ o\ A = A;$$
$$g_a\,(E\ o\ \ 0) = g_a\,(B) = B,\ \ g_a\,(E)\ \ o\ g_a\,(0) = F\ o\ A = B.$$

For other elements it can be proved that

$$g_a\,(L\ o\ M) = g_a\,(L)\ o\ g_a\,(M)\ \text{ for } L, M \in X.$$

So g_a is a homomorphism.

PARTIAL HOMOMORPHISM

Definition (3.1) : Let $(X;\ *,\ 1)$ be a BE–algebra and let $f : X \to X$. If there exists a subalgebra M of X such that

$$f(x * y) = f(x) * f(y)$$

for all $x, y \in M$ but $f\,(x * y) \neq f\,(x) * f\,(y)$ for some $x, y \in X$, then we say that f is a partial homomorphism on X with respect to subalgebra M.

Example (3.2) : Let $(X,\ T\ )$ be a topological space where T contains only clopen (closed and open) subsets. For $A, B \in T$ we define a binary operation 'o' on T as

$$A\ \ o\ B = A^c \cup B. \qquad \ldots (3.1)$$

Then $(T;\ o,\ 1)$ is a BE–algebra with zero element $0 \equiv \phi$ and $1 \equiv X$ [5].

For a fixed $a \in X$, let S be the collection of those elements of T which contain a. Now $L, M \in S \Rightarrow L\ o\ M = L^c \cup M \in S$, since $L^c \cup M$ contains a. So S is a BE–subalgebra of T.

Let $f_a : T \to T$ be a mapping defined as

$$f_a(L) = L \text{ or } L^c \text{ according } a \in L \text{ or } a \notin L.$$

Let $L, M \in S$, we have

$$f_a(L\ o\ M) = f_a(L^c \cup M) = L^c \cup M,$$

since $a \in M$. Also in this case

$$f_a(L) = L,\ f_a(M) = M \text{ and } f_a(L)\ o\ f_a(M) = L\ o\ M = L^c \cup M.$$

Thus $f_a(L\ o\ M) = f_a(L)\ o\ f_a(M)$ is satisfied for all $L, M \in S$.

Again, $a \notin L$ and $a \in M \Rightarrow f_a(L\ o\ M) = f_a(L^c \cup M) = L^c \cup M$, since $a \in L^c \cup M$.

In this case, $f_a(L) = L^c$ and $f_a(M) = M$ which gives $f_a(L)\ o\ f_a(M)$

$$= L^c\ o\ M = L \cup M.$$

So $f_a(L\ o\ M) \neq f_a(L)\ o\ f_a(M)$ for some $L, M \in T$.

This means that f_a is a partial homomorphism in T with respect to subalgebra S.

PAIR HOMOMORPHISM

Definition (4.1) : Let $(X; *, 1)$ and $(Y; o, e)$ be BE–algebras.

Let $f : X \to Y$ and $g : X \to Y$ be mappings. The pair (f, g) is said to be pair homomorphism if

$$f(x * y) = g(x)\ o\ g(y) \qquad \ldots (4.1)$$

for all $x, y \in X$.

Example (4.2): In the above definition, let $f(x) = e$ for all $x \in X$ and $g(t) = a \in Y$ for all $t \in X$, then the pair (f, g) is pair homomorphism. For,

$$f(x * y) = e = g(x)\ o\ g(y)$$

for all $x, y \in X$.

Note (4.3) : Pair (f, g) is pair homomorphism

$\not\Rightarrow$ Pair (g, f) is pair homomorphism.

In the above example $g(x * y) = a \neq e = f(x)\ o\ f(y)$.

Note (4.4) : If $g(1) = e$ then f coincides with g and f is a homomorphism. For $x \in X$ we have

$$\begin{aligned} f(x) &= f(1 * x) = g(1)\ o\ g(x) \\ &= e\ o\ g(x) \\ &= g(x). \end{aligned}$$

Note (4.5) : From the above example it also follows that in pair (f, g) g may not be unique for a mapping f.

Example (4.6) : Let $(X; *, 1)$ be a BE–algebra with zero element 0 and let $Y = X \times X$.

Then $(Y; \odot, (1, 1))$ is a BE–algebra [4, 2014] with zero element $(0, 0)$ where $\odot$ is defined as

$$(x_1, x_2) \odot (y_1, y_2) = (x_1 * y_1, x_2 * y_2).$$

We consider mappings $P_1, Q_1 : Y \to Y$

defined as $P_1(x_1, x_2) = (1, x_2)$ and $Q(x_1, x_2) = (0, x_2)$ for all $(x_1, x_2) \in Y$. Then

$$\begin{aligned} P_1((x_1, x_2) \odot (y_1, y_2)) &= P_1(x_1 * y_1, x_2 * y_2) \\ &= (1, x_2 * y_2) \\ &= (0, x_2) \odot (0, y_2) \\ &= Q_1(x_1, x_2) \odot Q_1(y_1, y_2) \end{aligned}$$

for all $(x_1, x_2), (y_1, y_2) \in Y$. So (P_1, Q_1) is a pair homomorphism.

Lemma (4.7) : Let $f, g : (X; *, 1) \to (Y; o, e)$ be a mapping such that pair (f, g) is a homomorphism. Then

$$f(1) = e;$$

$$x \le y \Rightarrow g(x) \le g(y);$$

Proof: (a) We have $f(1) = f(1 * 1) = g(1) \; o \; g(1) = e$.

(b) Also $x \le y \Rightarrow x * y = 1$

$\Rightarrow f(x * y) = f(1) = e$

$\Rightarrow g(x) \; o \; g(y) = e$

$\Rightarrow g(x) \le g(y)$.

Definition (4.8) : Let (f, g) be a pair homomorphism from a BE– algebra $(X; *, 1)$ into a BE–algebra $(Y; o, e)$. Then Ker f is defined as $\text{Ker} f = \{x \in X : f(x) = e\}$.

Theorem (4.9) : Let $(X; *, 1)$ and $(Y; o, e)$ be BE–algebra and let $f, g : X \to Y$. Let (f, g) be a pair homomorphism. Then

(a) $x \in \text{Ker} f$ iff $g(x) = g(1)$;

(b) Ker f is a BE–sub algebra of $(X; *, 1)$.

Proof: (a) Let (f, g) be a pair homomorphism and let $x \in \text{Ker} f$.

Then $e = f(x) = f(1 * x) = g(1) \; o \; g(x)$. This implies $g(1) \le g(x)$.

Again from lemma (4.7) (b) it follows that

$$x \le 1 \Rightarrow g(x) \le g(1). \text{ So } g(x) = g(1).$$

Conversely, suppose that

$$g(x) = g(1)$$

Then
$$\begin{aligned} f(x) &= f(1 * x) = g(1) \; o \; g(x) \\ &= g(1) \; o \; g(1) \\ &= e. \end{aligned}$$

So $x \in \text{Ker} f$.

(b) Let $x, y \in \text{Ker} f$. Then from above $g(x) = g(y) = g(1)$. Also $f(x * y) = g(x) \; o \; g(y) = g(1) \; o \; g(1) = e$. So $x * y \in \text{Ker} f$. This proves that Ker f is a BE–algebra.

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