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ξ-NORMAL SPACES

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The aim of this paper is to introduce a new class of normal spaces called ξ-normal spaces by utilizing ξ-open sets due to R. Devi *et al.* [1] and obtained several properties of such a space. Moreover, we obtain some new characterizations of ξ-normal spaces and preservation theorems.

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KEY WORDS AND PHRASES : ξ, ξ*, ξ**, *g*α, *rg*α-closed sets, ξ, ξ*, ξ**, *g*α, *rg*α-open sets, almost *g*α, *rg*α-functions, ξ-normal spaces.

INTRODUCTION

The aim of this paper is to introduce a new class of normal spaces called ξ-normal spaces by utilizing ξ-open sets and obtained several properties of such a space. R. Devi *et al.* [1] introduced the concept of ξ, ξ*, ξ**-closed sets and discuss some of their basic properties. Recently, Govindappa Navalagi *et al.* [2] introduced the concept of new normality axioms in topological spaces called (*sp*, *p*)-normal spaces, (*p*, *sp*)-normal spaces and (*gsp*, *sp*)-normal spaces by using pre-open, semi-preopen and *gsp*-open sets and obtained some weaker forms of regularity axioms are defined to the study the basic properties of these new normality axioms. Throughout this paper, (X, τ), (Y, σ) spaces always mean topological spaces X, Y respectively on which no separation axioms are assumed unless explicitly stated.

PRELIMINARIES

2.1. Definition. A subset A of a topological space X is called

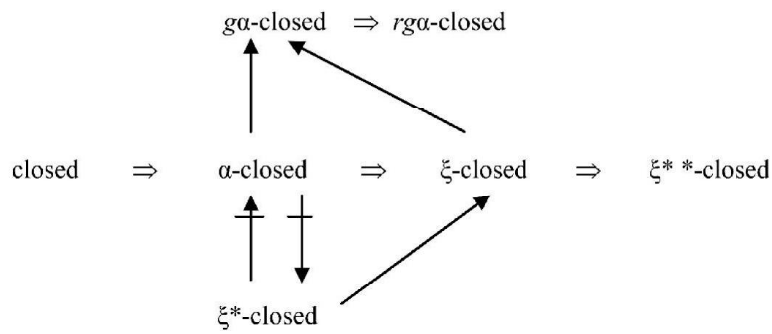
1. **regular closed** [5] if $A = \text{cl}(\text{int}(A))$.
2. **α-closed** [4] if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$.
3. ***g*α-closed** [3] if $\alpha\text{-cl}(A) \subseteq U$ whenever $A \subseteq U$, and U is an α-open in X.
4. **ξ-closed** [1] if $\alpha\text{-cl}(A) \subseteq U$ whenever $A \subseteq U$, and U is a $g\alpha$-open in X.
5. **ξ*-closed** [1] if $\alpha\text{-cl}(A) \subseteq \text{int}(U)$ whenever $A \subseteq U$, and U is a $g\alpha$-open in X.
6. **ξ**-closed** [1] if $\alpha\text{-cl}(A) \subseteq \text{int cl}(U)$ whenever $A \subseteq U$, and U is a $g\alpha$-open in X.
7. **regular α-open** [6] if there is a regular open set U such that $U \subseteq A \subseteq \alpha\text{-cl}(U)$.

8. ***rgα*-closed** [6] if α-cl (A) ⊂ U whenever $A \subset U$ and U is regularly α-open in X.

The complement of regular closed (resp. α-closed, *g*α-closed, ξ-cosed, ξ*-closed, ξ**-closed, *rg*α-closed, regular α-open) set is said to be **regular open** (resp. **α-open**, ***g*α-open**, **ξ-open**, **ξ*-open**, **ξ**-open**, ***rg*α-open**, **regularly α-closed**) set. The intersection of all α-closed subset of X containing A is called the **α-closure** (resp. **ξ-closure**) **of** ***A*** and is denoted by **α-cl (*A*)** (resp. ξ **–cl** (***A***)). The union of all α-open sets contained in A is called **α-interior** (resp. **ξ-interior**) **of** ***A*** and is denoted by **α-int (*A*)** (resp. **ξ-int (*A*)**. The family of α-open (resp. ξ-open, regularly open) sets of a space X is denoted by ***αO* (*X*)** (resp. **ξ*O* (*X*)**, ***RO* (*X*)**). The family of α-closed (resp. ξ-closed) sets of a space X is denoted by ***αC* (*X*)** (resp. **ξ*C* (*X*)**).

2.2. Remark. Every α-closed (resp. α-open) set is ξ-closed (resp. ξ-open) set.

Definitions stated above, we have the following diagram:



However the converses of the above are not true may be seen by the following examples.

2.3. Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then the set $A = \{c\}$ is α-closed as well as ξ, ξ**-closed but not closed in X.

2.4. Example. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}\}$. Then the set $A = \{b\}$ is ξ-closed but not ξ*-closed. Then the set $A = \{a\}$ is ξ**-closed but not ξ-closed in X.

2.5. Example. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{b\}, \{a, c\}, X\}$. Then the set $A = \{a\}$ is *g*α-closed but not ξ-closed in X.

2.6. Example. Let $X = \{a, b, c, d, e\}$ and $\tau = \{\phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}, X\}$. Then the set $A = \{a, d, e\}$ is *rg*α-closed but not *g*α-closed in X.

2.7. Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then the set $A = \{c\}$ is *g*α-closed but not closed in X.

The following examples show that α-closedness and ξ**-closedness are independent.

2.8. Example. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}\}$. Then the set $A = \{b\}$ is α-closed but not ξ*-closed in X.

2.9. Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{c, d\}, X\}$. Indeed $\tau^\alpha = \{\phi, \{c, d\}, \{b, c, d\}, \{a, c, d\}, X\}$. Then the set $A = \{a, b, c\}$ is ξ-closed; it is neither closed nor α-closed. Then the set $A = \{a, b, c\}$ is ξ*-closed but not α-closed in X.

ξ-NORMAL SPACES

3.1. Definition. A topological space X is said to be normal (resp. ξ-normal, ξ^{**}-normal) if for every pair of disjoint closed sets A and B, there exist open (resp. ξ-open, ξ^{**}-open) sets U and V such that $A \subset U$ and $B \subset V$.

3.2. Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $A = \{a\}$ and $B = \phi$ are disjoint closed sets, there exist disjoint open sets $U = \{a, c, d\}$ and $V = \{b\}$ such that $A \subset U$ and $B \subset V$. Hence X is normal as well as ξ- normal, ξ^{**}- normal because every open set is ξ-open as well as ξ^{**}-open set.

3.3. Remark. By the definitions and examples stated above, we have the following diagram:

$$\text{normal} \Rightarrow \alpha\text{-normal} \Rightarrow \xi\text{-normal} \Rightarrow \xi^{**}\text{-normal}.$$

3.5. Lemma. A subset A of a topological space X is $rg\alpha$-open iff $F \subset \alpha\text{-int}(A)$ whenever F is regularly closed and $F \subset A$.

3.6. Definition. A function $f : X \rightarrow Y$ is said to be **almost $g\alpha$-closed** (resp. **almost $rg\alpha$-closed**) if for every regular closed set F of X, $f(F)$ is $g\alpha$-closed (resp. $rg\alpha$-closed) in Y.

3.7. Theorem. A surjection $f : X \rightarrow Y$ is a almost $rg\alpha$-closed if and only if for each subsets S of Y and each $U \in RO(X)$ containing $f^{-1}(S)$ there exists a $rg\alpha$-open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof. Necessity. Suppose that f is almost $rg\alpha$-closed. Let S be a subset of Y and $U \in RO(X)$ containing $f^{-1}(S)$. If $V = Y - f(X - U)$, then V is a $rg\alpha$-open set of Y such that $S \subset V$ and $f^{1}(V) \subset U$.

Sufficiency. Let F be any regular closed set of X. Then $f^{-1}(Y - f(F)) \subset X - F$ and $X - F \in RO(X)$. There exists a $rg\alpha$-open set V of Y such that $Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Therefore, we have $f(F) \supset Y - V$ and $F \subset X - f^{-1}(V) \subset f^{-1}(Y - V)$. Hence we obtain $f(F) = Y - V$ and $f(F)$ is $rg\alpha$-closed in Y. This shows that f is almost $rg\alpha$-closed.

3.8. Theorem. For a topological space X, the following are equivalent :

(a) X is ξ-normal.

(b) For any disjoint closed sets A and B, there exist disjoint $g\alpha$-open sets U and V such that $A \subset U$ and $B \subset V$.

(c) For any disjoint closed sets A and B, there exist disjoint $rg\alpha$-open sets U and V such that $A \subset U$ and $B \subset V$.

(d) For any closed sets A and any open set B containing A, there exists $g\alpha$-open set U of X such that $A \subset U \subset \alpha\text{-cl}(U) \subset B$.

(e) For any closed sets A and any open set B containing A, there exists $rg\alpha$-open set U of X such that $A \subset U \subset \alpha\text{-cl}(U) \subset B$.

Proof. (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c) , (d) $\Rightarrow$ (e), (c) $\Rightarrow$ (d) and (e) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). Let X be a ξ-normal. Let A, B be disjoint closed sets in X. By assumption, there exist disjoint ξ-open sets U and V such that $A \subset U$ and $B \subset V$. Since every ξ-open set is $g\alpha$-open set, U, V are $g\alpha$-open sets such that $A \subset U$ and $B \subset V$.

(b) $\Rightarrow$ (c). Let A, B be disjoint closed sets in X. By assumption, there exist $g\alpha$-open sets U and V such that $A \subset U$ and $B \subset V$. Since every $g\alpha$-open set is $rg\alpha$-open set, U, V are $rg\alpha$-open sets such that $A \subset U$ and $B \subset V$.

(d) $\Rightarrow$ (e). Let A be any closed set and B be any open set containing A. By assumption, there exists $g\alpha$-open set U of X such that $A \subset U \subset \alpha\text{-cl}(U) \subset B$. Since every $g\alpha$-open set is $rg\alpha$-open set, there exists $rg\alpha$-open set U of X such that $A \subset U \subset \alpha\text{-cl}(U) \subset B$.

(c) $\Rightarrow$ (d). Let A be any closed set and B be a open set containing A. By assumption, there exist $rg\alpha$-open set U and W such that $A \subset U$ and $X - B \subset W$. By **Lemma 3.5**, we get, $X - B \subset \alpha\text{-int}(W)$ and $\alpha\text{-cl}(U) \cap \alpha\text{-int}(W) = \phi$. Hence,

$$A \subset U \subset \alpha\text{-cl}(U) \subset X - \alpha\text{-int}(W) \subset B.$$

(e) $\Rightarrow$ (a). If A and B be any two disjoint closed sets of X. Then $A \subset X - B$ and $X - B$ is a open. By assumption, there exists $rg\alpha$-open set G of X such that $A \subset G \subset \alpha\text{-cl}(G) \subset X - B$. Put $U = \alpha\text{-int}(G)$, $V = X - \alpha\text{-cl}(G)$. Then U and V are disjoint ξ-open sets of X such that $A \subset U$ and $B \subset V$. Therefore, X is ξ-normal.

3.9. Theorem. If $f : X \rightarrow Y$ is a continuous almost ξ-closed surjection and X is ξ-normal space, then Y is ξ-normal.

Proof. Let A and B be any closed set and B be a open set containing A. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X. Since X is ξ-normal, there exists disjoint ξ-open set U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Let $G = \text{int}(\text{cl}(\text{int}(U)))$ and $H = \text{int}(\text{cl}(\text{int}(V)))$, then G and H are disjoint regularly open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By **Theorem 3.7**, there exist disjoint $rg\alpha$-open sets K and L of Y such that, $A \subset K$ and $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L. It follows from **Theorem 3.7**, that Y is ξ-normal.

3.10. Corollary. If $f : X \rightarrow Y$ is a continuous, ξ-closed surjection and X is a ξ-normal space, then Y is ξ-normal.

Proof. Easy to verify.

3.11. Corollary. If $f : X \rightarrow Y$ is a continuous almost $g\alpha$-closed surjection and X is a ξ-normal space, then Y is ξ-normal.

Proof. Easy to verify.

3.12. Corollary. If $f : X \rightarrow Y$ is a continuous almost $rg\alpha$-closed surjection and X is a ξ-normal space, then Y is ξ-normal.

Proof. Easy to verify.

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