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EVEN AND EVEN SQUARE IDEALS OF A RING

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Following the notion of even and even square rings, here we introduce the notion of even and even square ideals of a ring R. We study some properties of these ideals and give some examples.

KEY-WORDS : Even square ring, even ideal, even square ideal.

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INTRODUCTION

This article deals with even and even square ideals of a ring *R*. The notion of even and even square rings has been introduced recently [1]. The notion of nil elements which are special type of nilpotent elements has been introduced in [1-2]. The idea to study even and even square ideals of a ring R has originated through the study of even square rings and related notions.

Recall that an element a of a ring R is called a nil element if $a^2 = 2a = 0.$ An element a of a ring R is called an even element if $a \in 2R$ and an element a of a ring R is called an even square element if $a^2 \in 2R.$. We shall use these definitions to give examples of even and even square ideals.

There are well established notion of various type of ideals in a ring *R* like prime ideal, maximal ideal, etc. [3-10]. In the case of prime or maximal ideals the whole ring is never a prime or maximal ideal.

However in the case of even and even square ideals the whole ring can be an even or even square ideal. It is seen that if a ring R has an even square ideal then R need not be an even square ring. It may be noted that each even ideal of a ring R is an even square ideal however each even square ideal is not necessarily an even ideal. We provide some results on even and even square ideals.

SOME RESULTS AND EXAMPLES

Following the definition of even and even square rings first of all we define even and even square ideals of a ring.

Definition 1. Let R be a ring. An ideal I of a ring R is called an even ideal of R if $a \in 2I, \forall a \in I.$

Definition 2. Let R be a ring. An ideal I of a ring R is called an even square ideal of R if $a^2 \in 2I, \forall a \in I.$

Proposition 1. Let R be a ring and I, J are any two even ideals of R then $I+J$ and IJ are even ideals of $R.$

Proof. Let R be a ring and I, J are any two even ideals of $R.$ Clearly $I+J$ is an ideal of $R.$ We have to prove that $I+J$ is an even ideal of $R.$ Let $x_i \in I, y_i \in J$ then $\sum_{i=1}^{n}(x_i + y_i) \in I+J.$ Clearly $x_i \in 2I, \forall x_i \in I$ and $y_i \in 2I, \forall y_i \in I.$ This gives that $\sum_{i=1}^{n}(x_i + y_i) \in 2(I+J).$ This implies that $I+J$ is an even ideal of $R.$ Similarly one can easily prove that IJ is an even ideal of $R.$

Proposition 2. Let R be a ring and I, J are any two even square ideals of R then $I+J$ and IJ are even square ideals of R provided R is a commutative ring.

Proof. Let R be a commutative ring and I, J are even square ideals of $R.$ We shall prove that IJ is an even square ideal of $R.$ Clearly IJ is an ideal of R [3]. Let $x_i \in I, y_i \in J$ then we see that $\sum_{i=1}^{n} x_i y_i \in IJ.$ We have $\left(\sum_{i=1}^{n} x_i y_i\right)^2 = \left(x_1^2 y_1^2 + \ldots x_n^2 y_n^2\right) \; +2\sum_{i,j=1}^{n}\left(x_i x_j\right) y_i y_j,$ $i \neq j.$ I and J are even square ideals therefore $x_i^2 \in 2I$ and $y_i^2 \in 2J$ for each $i = 1, 2, \ldots, n.$ This implies $\sum_{i=1}^{n} x_i^2 y_i^2 \in 2IJ.$ Also, $2\sum_{i,j=1}^{n}\left(x_i x_j\right) y_i y_j \in 2IJ.$ Thus $\left(\sum_{i=1}^{n} x_i y_i\right)^2 \in 2IJ.$ Hence IJ is an even square ideal. Similarly one can easily prove that $I+J$ is an even square ideal of $R.$

Proposition 3. Let R be a field of characteristic $\neq 2$ then both the ideals of R are even square ideals .

Remark 1. The proof follows from the fact that each field of characteristic $\neq 2$ is an even square ring.

Proposition 4. Let R be a ring and I be an even square ideal of R then the ring $\frac{R}{I}$ is not necessarily an even square ring.

Proof. We shall give an example to prove this result. Let R be the ring of integers. Then the set I of all even square integers is an even square ideal of $R.$ However the factor ring $\frac{R}{I}$ is not an even square ring.

Remark 2. The ring of integers is not an even square ring. But it has even square ideals. Thus if R be a ring and it has even square ideals then R is not necessarily an even square ring.

Proposition 5. Let R be a ring and I be an ideal of R then the factor ring $\frac{R}{I}$ is an even square ring provided R be an even square ring.

Proof. Let R be a ring and I be an ideal of $R.$ We have to prove that the factor ring $\frac{R}{I}$ is an even square ring. Let $a \in R.$ Clearly $a + I \in \frac{R}{I}.$ We have $(a+I)^2 = a^2 + I.$ If R is an even square ring, $a^2 \in 2R, \forall a \in R.$ This implies $(a+I)^2 \in 2\frac{R}{I}, \forall a \in R.$ Hence $\frac{R}{I}$ is an even square ring.

Proposition 6. Let R be a ring and I be an ideal of R then the factor ring $\frac{R}{I}$ is an even ring provided R is an even ring.

Proof. The proof is similar to the above.

Some Examples of Even Square Ideals

1. An ideal I of a ring R consisting of nil elements of R is an even square ideal of $R.$
2. Let I be an ideal consisting of even elements of a ring R then I is an even square ideal of $R.$ In this case the ideal is not necessarily an even ideal.
3. Let I be an ideal consisting of even square elements of a ring R then I is an even square ideal of $R.$
4. Let Z be the ring of integers then for each $m \in Z,$ $I = \{mx : x \in E\}$ gives an even square ideal of $Z.$ Here E is the set of all even square integers.
5. Let us define $I = \left\{ \frac{p}{q} : (p,q) = 1, q \neq 0, p \in E, q \in Z \right\}$ then it is easy to verify that I is an even square ideal of the ring of rational numbers.

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